"THE COMPUTABLE SURREAL NUMBERS" ①

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JOINT WORK WITH MYSELF & DAN TURETSKY, FOLLOWING UP DISCUSSION ON MATHOVERFLOW

TURNS OUT TO BE RE-DISCOVERY OF WORK OF JACOB LURIE, 1998.

- REVIEW OF SURREAL NUMBERS

(SEE MY ELEMENTARY INTRODUCTION ON INFINITELY MORE)

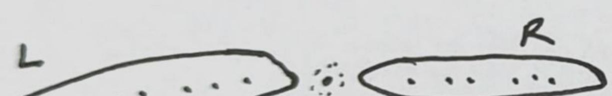
SURREALS ~~ARE~~ PROVIDE "UNIFICATION OF ALL NUMBERS GREAT & SMALL".

ORDINALS, REALS, INFINITESIMALS.

CONSTRUCTION PROCEEDS IN STAGES.

ITERATIVELY FILL ALL GAPS.

<u>SURREAL GENERATION RULE</u> AT EVERY STAGE, IN ALL POSSIBLE WAYS SEPARATE THE NUMBERS INTO A LOWER SET L AND UPPER SET R AND CREATE NEW SURREAL NUMBER $x = \{L \mid R\}$ STRICTLY BETWEEN.



$x = \{L \mid R\}$.

<u>SURREAL GENESIS</u>

- DAY 0. $0 = \{\ \mid\ \}$ IS BORN

1. $1 = \{0 \mid\ \}$ $-1 = \{\ \mid 0\}$

2

3

$\vdots$

ω $-\omega$ $-\epsilon$ ϵ e π $\sqrt{2}$ $\frac{1}{3}$ ω $\frac{p}{2^k} \pm \epsilon$

$\omega+1$

$\vdots$

0

$-1 \quad 0 \quad 1$

$-2 \quad -1 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad 2$

$$N^o = \bigcup_\alpha N^o_\alpha$$

SURREAL NUMERALS: WHENEVER L IS BELOW R, $\{L \mid R\}$ MEANS FIRST-BORN NUMBER IN THE GAP.

- ALTERNATIVE NUMERAL-BASED ACCT OF ORDER. $x \leq y$ IFF NO OBSTACLE I.E. $\neg\, y \leq x_L \wedge \neg\, y_R \leq x$.
- ALTERNATIVE SIGN SEQUENCE APPROACH. <u>TREE STRUCTURE.</u>

$\{+,-\}^{<\mathrm{ORD}}$.

$\langle\rangle = 0$

$\langle +\rangle = 1$

$\langle -\rangle = -1$

$\langle +-\rangle = \frac{1}{2}$

$\omega = \langle +++\cdots\rangle$

$\epsilon = \langle +---\cdots\rangle$

$\langle s\rangle = \{s \mid t\}_{t \sqsubseteq s}$

"THE COMPUTABLE SURREAL NUMBERS" ②

- SURREAL ORDER IS SET SATURATED
  - UNIVERSAL FOR ALL LINEAR ORDERS. (EVEN CLASS LINEAR ORDERS, IF GLOBAL CHOICE)
    OPEN: SHOW GLOBAL CHOICE IS REQUIRED.
- FUNDAMENTALLY DISCONTINUOUS — NO CONV. SEQ. EVERY SET IS DISCRETE.
- ADDITIVE GROUP STRUCTURE

$$x+y =_{DEF} \{ x_L + y \quad x+y_L \mid x+y_R \quad x_R + y \}.$$

$x+y=y+x$. ALSO ASSOCIATIVE ✓

CHECK $x+0 = \{x_L+0 \mid x_R+0\} = \{x_L \mid x_R\} = x$.

$$\tfrac{1}{2}+\tfrac{1}{2} = \{0 \mid 1\} + \{0 \mid 1\} = \{\tfrac{1}{2}+0,\ 0+\tfrac{1}{2} \mid 1+\tfrac{1}{2},\ \tfrac{1}{2}+1\} = 1.$$

$$1+1 = \{0 \mid \ \} + \{0 \mid \ \} = \{1+0,\ 0+1 \mid \ \} = \{1 \mid \ \} = 2$$

NEGATION $-x = \{-x_R \mid -x_L\}$ $\quad -0=0$, $--x = x$

$$x \le y \leftrightarrow -y \le -x$$
$$\leftrightarrow 0 \le y-x$$

$0 \le x-x \le 0$. $\quad x-x=0$.

~~SUBTRACTION~~

- MULTIPLICATION

NAIVE ATTEMPT: $x*y = \{ x*y_L,\ x_L*y \mid x*y_R,\ x_R*y\}$

DOESN'T WORK. $1*0 = \{0 \mid \ \} * \{ \mid \ \} = \{0*0 \mid \ \} = \{0 \mid \ \} = 1$.

RIGHT VERSION: $x \cdot y = \{ x_L y + x y_L - x_L y_L,\ x_R y + x y_R - x_R y_R \mid x_L y + x y_R - x_L y_R,\ x y_L + x_R y - x_R y_L \}$.

MOTIVATED BY: $0 < (x-x_L)(y-y_L) = x\cdot y - x_L y - x y_L + x_L y_L$.

$$x_L y + x y_L - x_L y_L < xy$$

ASSOCIATIVE
COMMUTATIVE
DISTRIBUTIVE ✓

$x \cdot 0 = \{ \mid \} = 0$. ✓

$x \cdot 1 = x \cdot \{0 \mid \ \} = \{x_L \cdot 0 + x\cdot 0 - x_L \cdot 0 \mid x\cdot 0 + x_R \cdot 1 - x_R \cdot 0\} = \{x_L \mid x_R\} = x$.

$2\cdot 2 = \cdots = 4$

$2\cdot 3 = \cdots = 6$

$$\varepsilon \cdot \omega = \{0 \mid \cdots \tfrac{1}{4}\ \tfrac{1}{2}\ 0\} \cdot \{0, 1, 2, \cdots \mid \ \}$$
$$= \{\varepsilon \cdot n \mid \varepsilon \cdot n + \tfrac{1}{2^n}\cdot\omega - n \cdot \tfrac{1}{2^n}\} = 1.$$

- MULTIPLICATIVE INVERSE

$$\frac{1}{y} = \left\{ 0,\ \frac{1+(y_R-y)\left(\frac{1}{y}\right)_L}{y_R},\ \frac{1+(y_L-y)\left(\frac{1}{y}\right)_R}{y_L} \;\middle|\; \frac{1+(y_L-y)\left(\frac{1}{y}\right)_L}{y_L},\ \frac{1+(y_R-y)\left(\frac{1}{y}\right)_R}{y_R} \right\}$$

GENERATED ITERATIVELY, BUT ONLY FOR POSITIVE y, POSITIVE y_L. IGNORE NONPOSITIVE TERMS

- ROOTS

$$\sqrt{x} = \left\{ \sqrt{x_L},\ \frac{x+(\sqrt{x})_L(\sqrt{x})_R}{(\sqrt{x})_L+(\sqrt{x})_R} \;\middle|\; \sqrt{x^R},\ \frac{x+(\sqrt{x})_L(\sqrt{x})_L^*}{(\sqrt{x})_L+(\sqrt{x})_L^*},\ \frac{x+(\sqrt{x})_R(\sqrt{x})_R^*}{(\sqrt{x})_R+(\sqrt{x})_R^*} \right\}.$$

$x > 0$.

- SURREALS No FORM A SATURATED REAL-CLOSED FIELD.
  SURREALS AS HYPERREALS. NONSTANDARD ANALYSIS. CALCULUS.

• CHALLENGE QUESTIONS (FOR THE GRADUATE STUDENTS)

1. HOW MANY DISJOINT INTERVALS FIT IN $[0, \omega]$?
2. SURREAL HEINE-BOREL THEOREM? I.E. IS $[0,1]$ COMPACT?
   A: CLASS COVER WITH ε INTERVALS — NO SET COVER.
   BUT EVERY SET COVER BY INTERVALS ADMITS A FINITE SUBCOVER
3. AUTOMORPHISMS OF No?
   CAN YOU MOVE ω TO $\omega+1$? $\omega+\sqrt{\pi}$ ω_1
   MOVE π TO $\pi+\varepsilon$
   GLOBAL CHOICE ISSUE
   DON'T ACTUALLY NEED GLOBAL CHOICE IN BACK-AND-FORTH ARG.
4. IS $\{L \mid R\}$ STRUCTURAL?
   A: NO!

• UNIVERSALITY

EVERY CLASS REAL-CLOSED FIELD EMBEDS INTO No.
LINEAR ORDER
SEEMS TO NEED GLOBAL CHOICE.

QUESTION (JDH) DOES ZFC PROVE THAT (No, <) IS UNIVERSAL FOR ALL CLASS ORDERS?

"THE COMPUTABLE SURREAL NUMBERS" 1

COMPUTABLE SURREAL NUMBERS   MAIN TOPIC

DEFINITION P IS A COMP SURREAL PROGRAM WITH VALUE α_p IFF

P ENUMERATES SETS L_p, R_p OF SURREAL PROGRAMS AND

$$\alpha_p = \{ \alpha_q \mid \alpha_r \}_{\substack{q \in L_p \\ r \in R_p}}.$$

WELL-FOUNDED RECURSIVE DEFN, SIMILAR TO KLEENE'S $\mathcal{O}$.

EXAMPLES $0 = \{ \mid \}$.

1

2

$\frac{1}{2}$

ETC.

EVERY COMPUTABLE REAL NUMBER

COMPUTABLE ORDINAL

NATURAL QUESTIONS (i) ARE COMP'BLE SURREALS A REAL-CLOSED FIELD?

(ii) WHICH REALS ARE COMPUTABLE SURREALS?

(iii) COMPLEXITY OF $X < Y$

BIRTHDAYS

BINARY REP

~~OBSERVATION~~ cN^o IS AN ADDITIVE GROUP.

$$X + Y = \{ X + Y_L,\ X_L + Y \mid X + Y_R,\ X_R + Y \}.$$

+ IS COMPUTABLE BY KLEENE RECURSION THM.

RING MULTIPLICATION IS COMPUTABLE

$$X \cdot Y = \{ X_L Y + X Y_L - X_L Y_L,\ X_R Y + X Y_R - X_R Y_R \mid X_L Y + X Y_R - X_L Y_R,\ X Y_L + X_R Y - X_R Y_L \}.$$

USE KLEENE RECURSION TO KNOW THAT $\cdot$ IS COMPUTABLE.

FIELD? PROBLEM WITH $\frac{1}{Y}$ NEEDED TO CHECK FOR POSITIVE TERMS.

BUT $X < Y$ DOES NOT SEEM COMPUTABLE.

THEOREM SOME COMPUTABLE SURREAL REAL NUMBERS ARE NOT COMPUTABLE REAL NUMBERS.

PF: FOR ANY TM PROGRAM e, DEFINE h_e IN $c\mathbb{N}^\mathbb{Q}$ s.t. $\alpha_{h_e}=0$ OR $\alpha_{h_e}=1$

$\alpha_{h_e}=1 \longleftrightarrow \varphi_e(e)\downarrow$.

h_e STARTS WITH $\{\, 1 \,\}$. SIMULATES $\varphi_e(e)$. IF HALTS, PUT 0 INTO L.

LET $h = \left\{ \sum_{e \le n} h_e \cdot 2^{-e} \;\middle|\; \left(\sum_{e\le n} h_e \cdot 2^{-e}\right) + \frac{1}{2^n} \right\}_n$

SO $\alpha_h = \sum_{\varphi_e(e)\downarrow} \frac{1}{2^e}$ NOT COMPUTABLE REAL

IS COMPUTABLE SURREAL. $\boxtimes$

KEY POINT:
CAN COMPUTE WITH h_e
EVEN THOUGH WE DON'T
KNOW THE VALUE.

THM c.e. REALS ARE COMPUTABLE SURREAL.

PF: ASSUME x IS LEFT c.e. $x = N + \sum_{k\in X} 2^{-k}$ X c.e.

LET $r_k = \begin{cases} 0 & k \notin X \\ 1 & k\in X \end{cases}$ COMPUTABLE SURREAL.

DEFINE r TO PLACE $N + \sum_{k\le n} r_k \cdot 2^{-k}$ INTO L.

$N + \left(\sum_{k\le n} r_k 2^{-k}\right) + \frac{1}{2^n}$ INTO R.

$\alpha_r = x$. $\boxtimes$

ALSO GET RIGHT c.e., d.c.e, ETC.

THM FOR EVERY ARITHMETIC SENTENCE φ, $\exists P_\varphi$ $\quad \alpha_{P_\varphi} = \begin{cases} 1 & \varphi \text{ HOLDS} \\ 0 & \text{o.w.} \end{cases}$

UNIFORMLY COMP'BLE $\varphi \mapsto P_\varphi$.

PF: TRUE FOR ATOMIC φ.

$P_{\neg\varphi} = 1 - P_\varphi$.

$P_{\varphi\wedge\psi} = P_\varphi \cdot P_\psi$

$P_{\exists x\, \varphi(x)} \sim \left\{ \underbrace{P_{\varphi(x)} - 1}_{\substack{=-1 \text{ IF } \varphi(x) \text{ FALSE} \\ =0 \text{ IF } \varphi(x) \text{ TRUE}}} \;\middle|\; \right\}_x$

CONCLUSION EVERY ARITHMETICALLY DEFINABLE REAL IS COMPUTABLE SURREAL.

PUSH INTO HYPERARITHMETIC.

THEOREM FOR EVERY EFFECTIVE $\varphi \in L_{\omega_1,\omega}$ THERE IS P_φ WITH $\alpha_{P_\varphi} = \begin{cases} 1 & \varphi \\ 0 & \neg\varphi \end{cases}$

UNIFORMLY COMPUTABLE $\varphi \mapsto P_\varphi$.

POINT: $\varphi = \bigvee_n \varphi_n \qquad P_\varphi = \{ P_{\varphi_n} - 1 \mid \quad \}_n$

CONCLUSION EVERY HYPERARITHMETIC REAL IS COMPUTABLE SURREAL.

THEOREM THE COMPUTABLE SURREALS ARE EXACTLY THOSE WITH A HYPERARITHMETIC +- SEQUENCE.

COROLLARY COMPUTABLE SURREALS ARE A FIELD. REAL-CLOSED FIELD.

WE COULDN'T SEE DIRECTLY, EARLIER, THAT $\frac{1}{y}$ IS COMPUTABLE.

BUT IT IS HYPERARITHMETIC, AND THAT IS GOOD ENOUGH BY PREVIOUS.

SO WE KNOW $\frac{1}{y}$ IS COMPUTABLE, ROOTS ETC.

THEOREM $cN^o = N^o \cap L_{\omega_1^{ck}}$.