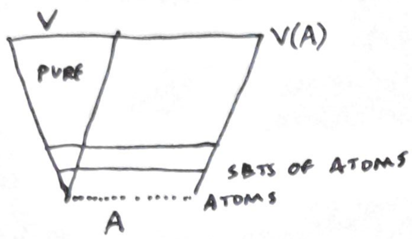
"SET THEORY WITH ABUNDANT URELEMENTS" ①

STUK 10, OXFORD 14 JUNE 2023

- JOINT WORK WITH BOKAI YAO
- SET THEORY WITH URELEMENTS



- FORM SETS OVER A CLASS OF ALREADY EXISTING PRIMITIVE OBJECTS
- e.g. NUMBERS, POINTS, IRREDUCIBLE MATH'L OBJS.

$\langle V(A), \in, A \rangle$

↑ URELEMENT PREDICATE $\emptyset$ VS. ATOMS

- AXIOMS
  - EXTENSIONALITY
  - ZFCU = EXT, FOUND, PAIR, UNION, POWER, ∞, SEP, COLL ←, CHOICE
    - TAKE CARE
      - A HIERARCHY OF THEORIES
      - EXPECTED EQUIVS. BREAK
  - URELT. SUPPORT OF u KER(u) = SET OF ATOMS IN TC(u).
    - ~~[illegible]~~
      - $V(B)$ FOR $B \subseteq A$ $\models$ ZFCU
      - $V(A) = \bigcup_{w \subseteq A} V(w)$
      - $V_\alpha(A)$ NOT SET-LIKE IF A PROPER CLASS.
  - NONRIGIDITY
    - IN ZFC, TRANSITIVE SETS ARE RIGID. V IS RIGID. $\pi : (X, \in) \cong (Y, \in)$ $\pi(x) = \{\pi(y) \mid y \in x\}$.
    - BUT NOT WITH URELTS.
    - EVERY PERMUTATION $\pi : A \to A$ EXTENDS TO AUTO. $\pi : V(A) \cong V(A)$.
    - IMPACTS REPLACEMENT.

$W = \bigcup_{w \subseteq A \text{ finite}} V(w)$ FINITELY-SUPPORTED SETS. $\models$ REPLACEMENT ZFCU_R

$\not\models$ COLLECTION

- ATOMS ARE PROPER CLASS
- BUT EVERY SET OF ATOMS IS FINITE

COMPARE ZFC^-

SIMILAR: $\bar{W}$ = CTBLY SUPPORTED SETS. GET COLLECTION. ZFCU.

- A PROPER CLASS
- BUT EVERY SET OF ATOMS IS CTBLE

ω_1-DC SCHEME FAILS.

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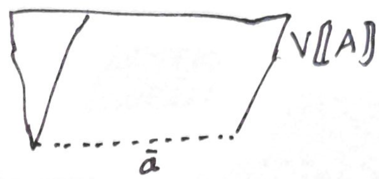
- INTERPRETING URELEMENT SET THEORY IN ZFC

GOAL: ZFCU + "THERE ARE ORD MANY ATOMS"
$\quad\quad\quad\quad\quad\quad V$ MANY
$\quad\quad\quad\quad\quad\quad A$ MANY $\quad\quad$ ARB. CLASS A.

START IN $V \models ZFC$. CLASS A.

MAKE A COPY OF A.

$\bar{a} = \langle 0, a \rangle$ FOR $a \in A$.



PLACE INTO $V[\![A]\!]$

CLOSE UNDER: $y \subseteq V[\![A]\!] \rightarrow \bar{y} = \langle 1, y \rangle \in V[\![A]\!]$.

$x \mathrel{\bar{\in}} \bar{y}$ IFF $x \in y$. $\quad\quad$ $\langle 0, a \rangle$ ATOMS
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\langle 1, y \rangle$ SETS.

$\langle V[\![A]\!], \bar{\in}, A \rangle \models ZFCU$. $\quad\quad$ $\check{u} = \langle 1, \{\check{v} \mid v \in u\} \rangle$.

$\vec{A}$ = ATOM ENUMERATION PREDICATE. $\check{a} \mapsto \bar{a}$.

__THM__ $\langle V, \in \rangle$ BI-INTERPRETABLE WITH $\langle V[\![A]\!], \bar{\in}, \vec{A} \rangle \models ZFCU$ + "A MANY ATOMS".

NOTE: $V[\![A]\!] \models$ EVERY SET IS EQUINUM. WITH A PURE SET.
- PHILOSOPHICALLY IMPORTANT
- STRUCTURALISM.

__COROLLARY__ FOLLOWING THEORIES BI-INTERPRETABLE

ZFC
$ZFC\vec{U}$ + "$\exists\ \omega$ MANY ATOMS"
$ZFC\vec{U}$ + "ORD MANY ATOMS"
$\quad\quad\quad\quad \mathbb{R}$
$\quad\quad\quad\quad V$
$\quad\quad\quad\quad A \quad\quad$ ANY DEFINABLE CLASS A.

$ZFC\vec{U}$ IS NOT TIGHT.

__THEOREM__ FOLLOWING THEORIES ARE MUTUALLY INTERPRETABLE, BUT NOT BI-INTERPRETABLE:

ZFC
ZFCU
ZFCU + "PROPER CLASS OF ATOMS"

PROOF: MUTUALLY INTERPRETABLE. ✓
NOT BI. ~~[illegible]~~ NON-RIGIDITY.

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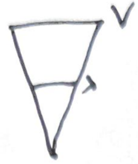
- REFLECTION

LÉVY-MONTAGUE



V REFLECTS TO V_λ.

CLUB OF Σ_n-CORRECT CARDINALS.

IN URELEMENT CONTEXT,

DON'T WANT $V_\lambda(A)$. NOT A SET.

HAS HUGE SETS OF ATOMS, ABOVE λ.

<u>FIRST-ORDER REFLECTION</u> $\forall p\ \exists v \ni p$ φ IS ABS. FROM $V(A)$ TO V.

IF A IS A <u>SET</u> OF ATOMS, CAN STRATIFY $V(A)$ BY $V_\alpha(A)$ & PROVE REFLECTION.

IN FACT, CAN PROVE REFLECTION IN ZFCU GENERALLY.

- USES AC.
- ONLY NEED: EVERY SET IS EQUINUM. WITH A PURE SET.
- ACTUALLY ONLY: PURE RELATIVE TO SOME SET u.

- CLASS THEORY

GÖDEL-BERNAYS $\langle M, \in^M, \mathcal{M} \rangle$ ↖ COLLECTION OF CLASSES. $(M, \in^M) \models ZFC$

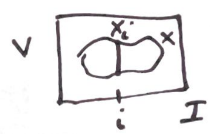
ALLOW CLASSES IN SEP, COLL. SCHEMES
COMP.

GLOBAL CHOICE.

KELLEY-MORSE

ALLOW 2^{ND} ORDER φ IN SCHEMES.

CC. CLASS CHOICE. $\forall i \in I\ \exists x\ \varphi(i,x,I) \rightarrow \exists X \subseteq I \times V\ \forall i\ \exists \varphi(i, X_i, I)$.



NOT PROVABLE IN KM.

<u>THM</u> (MAREK, MOSTOWSKI) KM+CC BI-INTERPRETABLE WITH $ZFC^- + \exists$ LARGEST CARD κ
1975 WILLIAMS '18. WHICH IS INACC.

PF: <u>UNROLLING</u> CONSTRUCTION.

SIMILAR TO CODING CTBLE SETS WITH REALS. e.g. $u \in H_{\omega_1}$ $(TC(\{u\}), \in) \cong (\mathbb{N}, E)$
SOME $E \subseteq \mathbb{N} \times \mathbb{N}$.

SIMILAR, REPRESENT META-SETS WITH ~~ORD~~ (ORD, E)

- WELL-FOUNDED
- EXTENSIONAL
- LARGEST ELT.

NEED KM TO IDENTIFY $\equiv$. NEED CC TO VALIDATE <u>COLLECTION</u>.

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- CLASS THEORY WITH URELEMENTS

GBCU $\langle M, \in^M, A, \mathcal{M} \rangle$

↑ CLASSES

↑ URELT. PREDICATE

WARNING: VARIATIONS OF GLOBAL AC NO LONGER EQUIV.

- CLASS CHOICE FUNCTION
- GLOBAL W.O.
- O.T. ORD.

- INTERPRETATIONS

IN $\langle V, \in, \mathcal{V} \rangle$ INTERPRET $\langle V[\![A]\!], \bar{\in}, \vec{A}, \mathcal{V}[\![A]\!] \rangle \models$ GBCU + "A MANY ATOMS".

<u>THM</u> FOLLOWING THEORIES ARE BI-INTERPRETABLE W/ PARS.

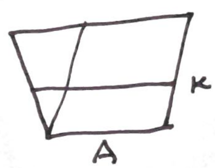
GBC
GBCU + "ω MANY ATOMS"
ORD
V MANY
ETC.

ALSO WITH KM, KMU + "ORD MANY ATOMS" ETC.

<u>QUESTION</u> WHAT IF MORE THAN ORD MANY ATOMS?

- ABUNDANT ATOMS $\langle V(A), \in, \mathcal{V} \rangle$

CONSIDER $V(A) \models$ GBCU + κ INACC.



CONSIDER $\langle H_\kappa(A), \in, \mathcal{H} \rangle$

↑ CLASSES $X \subseteq H_\kappa(A)$.

NOTE: A IS A CLASS, BUT MUCH LARGER THAN κ.

IF A HAS SIZE ORD.

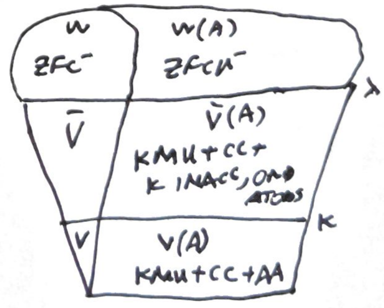
"SMALL" CLASS $= < A$.

<u>ABUNDANT ATOM AXIOM</u>

- CLASS OF ATOMS IS STRICTLY LARGER THAN ORD
- EVERY SMALL CLASS OF ATOMS HAS A SMALL POWER CLASS.
- EVERY SMALL-INDEXED CLASS OF SMALL CLASSES IS SMALL.

<u>THEOREM</u> FOLLOWING THEORIES ARE BI-INTERPRETABLE.

KMU + CC + AA
KM + CC + INACC.
KMU + CC + INACC + "ORD MANY ATOMS"
ZFC$^-$ + $\kappa < \lambda$ BOTH INAC + λ IS LARGEST CARD.
ZFCU$^-$ + $\kappa < \lambda$ INACC + λ LARGEST + λ MANY ATOMS



"SET THEORY WITH ABUNDANT URELEMENTS"

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- SECOND-ORDER REFLECTION

$\langle V, \in, \mathcal{V}\rangle \models \varphi(X)$ SECOND-ORDER.

$\Rightarrow \exists \lambda \quad V_\lambda \models \varphi(X \cap V_\lambda)$.

EQUIVALENT TO SAY: $(V, \in \mathcal{V}) \models \varphi(X) \Rightarrow \exists$ TRANS. SET $\cancel{v} \models \varphi(X \cap \cancel{v})$.

I.E. IN $\langle V, \in, P(V)\rangle$.

e.g. φ MIGHT ASSERT GBC ITSELF. SO v WILL BE V_λ λ INACC.

THM ① KM + SECOND-ORDER REFL. $\rightarrow$ $\forall n \, \exists$ STAT PROPER CLASS OF Π^1_n-INDESCR. CARDS.

② κ MSBLE $\rightarrow$ $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models$ KM + CC + 2^{ND} ORDER REFL.

PF: FOR ② S'POZE $V_\kappa \models \varphi(X)$. $j: V \to M$. SO $M \models \exists \kappa < j(\kappa) \; V_\kappa \models \varphi(X)$. $j(X) \cap V_\kappa$

SO $\exists \lambda < \kappa \; V_\lambda \models \varphi(X \cap \lambda)$. ⊠

- REFLECTION ERASES DISTINCTION BETWEEN GB & KM.

THM ~~GBC + 2^{ND} ORDER REFL $\Leftrightarrow$ KM, CC~~

FOLLOWING THEORIES ARE IDENTICAL:

[GBc + 2^{ND} ORDER REFL
KM + CC + 2^{ND} ORDER REFL.

~~KE~~

SAME FOR

GBcU + 2^{ND} ORDER REFL.
KMU + CC + 2^{ND} ORDER REFL.

~~2^{ND} SECOND ORDER~~

THM (YAO) IF $\exists$ SUPERCOMPACT CARDINAL, THEN
$\exists$ MODEL OF KMU + 2^{ND}-ORDER REFL + MORE THAN ORD ATOMS.

Q: IS SUPERCOMPACTNESS NEEDED?

HE USED κ^+-S.C. κ TO BUILD THE MODEL. BUT SEEMED TOO STRONG?

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- REFLECTING CARDINALS

DEF κ IS <u>2ND ORDER REFLECTIVE</u> IF $\forall M \models \varphi,\ \kappa \subseteq M$ (φ SECOND-ORDER, LANG SIZE $<\kappa$)
$\exists m \prec M$ $m \cap \kappa \in \kappa$, $m \models \varphi$.

<u>λ-REFLECTIVE</u> FOR $|M| \le \lambda$.

FOR <u>Π^1_n-ASSERTIONS</u> RESTRICT φ TO Π^1_n.

MAGIDOR: <u>SMALLEST</u> SUPERCOMPACT κ HAS: EVERY Π^1_1 SENTENCE φ TRUE IN SOME M IS TRUE IN SMALL SUBSTRUCTURE.

<u>THEOREM</u> ① κ IS λ-S.C. $\Rightarrow$ κ IS 2ND ORDER λ-REFLECTIVE.

② κ IS $2^{\lambda^{<\kappa}}$-REFL. FOR Π^1_1 $\Rightarrow$ κ IS λ-SUPERCOMPACT.

PROOF SKETCH: ① $j: V \to M$ λ-S.C.

$M = \langle \lambda, R, f, \ldots \rangle$ $\quad j(M) = \langle j(\lambda), j(R), j(f), \ldots \rangle$.

$j''M = \langle j''\lambda, j|R, j(f), \ldots \rangle \prec j(M)$.

SMALLER THAN $j(\kappa)$.

STILL SATISFIES φ, SINCE $\cong M$.

② EXISTENCE OF NORMAL FINE MEASURE ON $P_\kappa\lambda$ IS EXPRESSIBLE IN $\langle H_{(\lambda^{<\kappa})^+}, \in, \kappa, \lambda \rangle$.

BUT IF $m \prec H_{(\lambda^{<\kappa})^+}$ CAN USE $s = m \cap \lambda$ AS BASE OF PRINCIPAL FILTER.

FULFILLS IT OVER m ITSELF. ⊠

<u>COROLLARY</u> κ IS 2ND ORDER REFL.

IFF κ IS SUPERCOMPACT

IFF κ IS 2ND ORDER REFL. FOR Π^1_1.

<u>REFINED VERSION</u> ASSUME $\lambda = \lambda^{<\kappa}$. TFAE

① κ IS λ-REFL. FOR Π^1_1

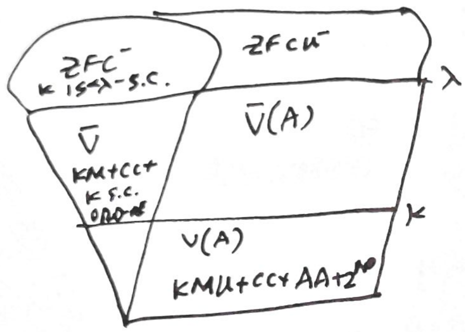
② κ IS NEARLY λ-S.C. $\quad$ SEE (SCHANKER)

<u>THM</u> IF κ IS λ-S.C., THEN IN $V[\![\lambda]\!]$ CONSIDER

$\langle H_\kappa(A), \in, \mathcal{U} \rangle \models$ KMU + CC + 2ND ORDER REFL, λ MANY ATOMS

<u>MAIN THEOREM</u> FOLLOWING ARE BI-INTERPRETABLE:

GBc + AA + 2ND ORDER REFL.
KMU + CC + AA + 2ND ORDER REFL.
KM + CC + κ IS S.C. + 2ND ORDER REFL.
KMU + CC + ORD ATOMS + κ IS S.C. + 2ND ORDER REFL.
ZFC^- + κ IS $<\lambda$-S.C. + 2ND ORDER λ-REFL. + λ IS LARGEST CARD. & INACC.
$ZFCU^-$ + λ ATOMS + κ IS $<\lambda$-S.C. + 2ND ORDER λ-REFL., λ LARG. CARD & INAC.



- PHILOSOPHICAL REMARKS

HISTORICALLY, URELTS WERE ABANDONED. BUT: SHALL WE ADMIT WEIRD SETS OF ATOMS?

MY VIEW: ON GROUNDS OF STRUCTURALISM.

MAIN LESSON OF MAIN THM:

IF YOU HAVE MORE THAN ORD ATOMS, YOU DIDN'T BUILD V HIGH ENOUGH.