Can there be natural instances of nonlinearity in the hierarchy of consistency strength?

Joel David Hamkins
Professor of Logic
Sir Peter Strawson Fellow

University of Oxford
University College, Oxford

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The linearity phenomenon

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Why should it be linear?

Significance of linearity

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Is the phenomenon directing us along the “one road upward” (Steel 2013), toward the final, ultimate mathematical truth?

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Many set theorists claim that amongst the *natural* assertions, consistency strengths are linearly ordered and indeed well ordered.

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Can we find natural instances of nonlinearity and illfoundedness in the hierarchy of consistency strength?

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Can we find natural instances of nonlinearity and illfoundedness in the hierarchy of consistency strength?

I shall try my best.

Consistency strength

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It is common to use PA as a base theory in the context of arithmetic and ZFC in a set-theoretic context.

But actually much weaker base theories usually suffice for most of the usual observations.

Formal instances of incomparability

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These will be the often-dismissed ‘unnatural’ instances of incomparability.

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σ asserts:

> *For any refutation of τ in* $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, *there is a smaller refutation of σ.*

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Each asserts that it has smallest refutation, if either is refutable.

Proof, continued.

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Since σ is not refutable, there is a model of $\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) + \sigma$, in which also this theory is inconsistent. So σ is refutable here in $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, but also true. So it has smallest refutation. This is provable in PA. So this model has $\mathrm{Con}(\mathrm{PA} + \sigma)$ and $\neg\,\mathrm{Con}(\mathrm{PA} + \tau)$.

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Similarly, since τ is not refutable, consider theory $\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) + \tau$ analogously. We find a model with $\mathrm{Con}(\mathrm{PA} + \tau)$ but $\neg\,\mathrm{Con}(\mathrm{PA} + \sigma)$.

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So σ,τ have incomparable consistency strength over PA. □

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The usual Rosser argument shows η is neither provable nor refutable in that theory.

Proof, continued.

Since η not provable, we get a model of $\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) + \neg\eta$. This model thinks η is provable with no smaller proof of $\neg\eta$. So $\mathrm{PA} \vdash \neg\eta$. So model thinks $\mathrm{Con}(\mathrm{PA} + \neg\eta) + \neg\,\mathrm{Con}(\mathrm{PA} + \eta)$.

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Since η not refutable, we get model of $\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) + \eta$, which thinks this theory inconsistent. Has proof of $\neg\eta$ from $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, so by η there is such proof with no smaller proof of η. So $\mathrm{PA} \vdash \eta$ and so this is a model of $\mathrm{Con}(\mathrm{PA} + \eta) + \neg\,\mathrm{Con}(\mathrm{PA} + \neg\eta)$.

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So neither $\mathrm{Con}(\mathrm{PA} + \eta)$ nor $\mathrm{Con}(\mathrm{PA} + \neg\eta)$ proves the other. So η and $\neg\eta$ have incomparable consistency strengths. □

Both η and $\neg\eta$ jump

Precisely because $\mathrm{PA} + \eta$ and $\mathrm{PA} + \neg\eta$ are incomparable in consistency strength, it follows that neither is equiconsistent with PA.

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This is an instance of *double-jumping* in consistency strength.

No jumping

A sentence ρ with $\mathrm{PA} + \rho$, $\mathrm{PA} + \neg\rho$ both equiconsistent with PA.
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Double jumping

A sentence such as η above, where both $\mathrm{PA} + \eta$ and $\mathrm{PA} + \neg\eta$ are strictly stronger than PA in consistency strength.

Consistency strengths are dense

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So $\mathrm{PA} + \mathrm{Con}(S) + \neg\,\mathrm{Con}(T)$ is consistent.

Let δ be the Rosser sentence of this theory.

Proof.

Since δ is not provable, get model of $\mathrm{Con}(S) + \neg\,\mathrm{Con}(T) + \neg\delta$. By $\neg\delta$, there is proof of δ with no smaller proof of $\neg\delta$. But PA can verify that, so $\mathrm{PA} \vdash \neg\delta$. So $\mathrm{Con}(S + \neg\delta) + \neg\,\mathrm{Con}(S + \delta) + \neg\,\mathrm{Con}(T)$.

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Let $U = T \vee (S + \delta)$ and $U' = T \vee (S + \neg\delta)$.

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So $\mathrm{Con}(U) = \mathrm{Con}(T) \vee \mathrm{Con}(S + \delta)$ and $\mathrm{Con}(U') = \mathrm{Con}(T) \vee \mathrm{Con}(S + \neg\delta)$.

The two models show U and U' are incomparable. Hence strictly intermediate between S and T. □

Natural instances of nonlinearity and ill-foundedness

I aim now at the hard task.

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To provide natural instances of nonlinear incomparability and ill-foundedness in the hierarchy of consistency strength, especially in the large cardinal hierarchy.

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As well as I am able...

Finitely many inaccessible cardinals

Consider the assertions that there are some finite number of inaccessible cardinals.

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Ultimately, we often describe a number n by specifying a computable procedure for computing it.

Nonlinearity in large cardinal assertions

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Indeed, there is a computable function f for which the assertions, "there are $f(n)$ many inaccessible cardinals," as n varies are incomparable and strongly independent with respect to consistency strength.

Strongly independent means that the consistency assertions freely generate the free countable Boolean algebra.

Universal computable function

The proof relies on an elementary version of the universal computable function.

Theorem

For any consistent $T \supseteq \mathrm{PA}$, there is a TM program e, such that for any function $f : \mathbb{N} \to \mathbb{N}$, there is model of T inside which program e on input n computes exactly $f(n)$.

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Thus, the program e can compute any desired function at all, if only you run the program in the right model.

Proof.

Program e searches for a proof from T of a statement of form:

> *"the function computed by e is not precisely given by this input/output list: $(k_0, n_0), \ldots, (k_r, n_r)$."*

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The point: you cannot refute any particular behavior for this function, since if you could, it would exhibit exactly that behavior.

Consequently, it is consistent that this program computes any desired particular values! □

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Inside that model, can find a model N where $\mathrm{ZFC} + f(n)$ many inaccessibles is consistent, but not $f(n) + 1$.

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Similar for any finite pattern of consistency. □

Pervasive natural instances of nonlinearity

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Pervasive natural instances of nonlinearity

The same argument works with Mahlo cardinals, measurable cardinals, and similarly with almost any of the usual large cardinal notions.

Thus, we find natural instances of incomparability for these kind of large cardinal existence assertions, throughout the large cardinal hierarchy.

A possible objection

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Amongst assertions of the form,

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Theorem

Amongst assertions of the form,

> *"There are as many supercompact cardinals as the running time of this specific computational process,"*

there are instances with incomparable consistency strength.

Indeed, there is a computable function f for which the supercompact running-time statements about $f(n)$ are incomparable and strongly independent with respect to consistency strength.

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Easily addressed.

Theorem

Amongst assertions of the form,

> *"There are as many supercompact cardinals as the running time of this specific computational process,"*

there are instances with incomparable consistency strength.

Indeed, there is a computable function f for which the supercompact running-time statements about $f(n)$ are incomparable and strongly independent with respect to consistency strength.

If these are a natural class of statements, then these are natural instances of incomparability.

More incomparability

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There is a program e for which "e halts on input n" are incomparable and strongly independent in consistency strength.

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2. By (1) there must be instances $n \notin A$ not provable in $\mathrm{PA} + \mathrm{Con}(T)$. So there is a model of $\mathrm{PA} + \mathrm{Con}(T) + n \in A$. But $n \in A$ implies $\mathrm{PA} \vdash n \in A$, so $\neg\,\mathrm{Con}(\mathrm{PA} + (n \notin A))$ here. So consistency strength of $n \notin A$ is not bounded by T. □

Consistency strength is inherent

Theorem

Let A be any computably enumerable m-complete decision problem.

1. *If A is PA-provably m-complete, with a provable reduction of the halting problem, then amongst true $n \notin A$, there are instances strictly exceeding any given consistency strength.*

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3. *Amongst consistent $n \in A$, there are instances of incomparable consistency strength and double-jumping strength.*
4. *Within both $n \notin A$, $n \in A$, there are effective enumerations that are strongly independent.*

Incomparable consistency strengths in tiling problem

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Are these natural instances of incomparability?

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We can similarly find incomparable strongly independent consistency strength for instances of:

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Are these not therefore natural instances of incomparability in consistency strength?

Incomparability in any c.e. undecidable set?

I had used m-completeness to find incomparability in decision problems A, but I suspect a more general result is possible.

Open Questions

1. Does every computably enumerable undecidable set A admit statements $n \notin A$ of incomparable consistency strength?

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Different cautious theory enumerations will pay attention to different specific indicators.

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The cautious enumeration ZFC° is sensible and realistic—it is what we would actually do when enumerating the theory.

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The cautious enumeration ZFC° *has all the same axioms as* ZFC*, but with strictly lower consistency strength.*

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So ZFC° has strictly weaker consistency strength than ZFC. □

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The proof is similar.

Ill-foundedness in consistency hierarchy

Iterating this idea, we find increasingly cautious enumerations of ZFC, strictly descending in consistency strength.

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So this is a natural instance of ill-foundedness in the hierarchy of consistency strength.

Cautious enumeration of ZFC plus an inaccessible

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The increasingly cautious enumerations constitute a natural instance of ill-foundedness:

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Theorem

The assertions above are incomparable and strongly independent in consistency strength. Each is strictly stronger than every I_n, but strictly weaker than I_ω.

Natural nonlinearity throughout

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This is what we would actually do.

What is “natural”?

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Mathematicians commonly adopt a *know-it-when-you-see-it* attitude to naturality.

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- Naturality talk is too often used to reject the unfamiliar.
- “Unnatural” solutions often indicate that a question wasn’t well formulated.
- We may legitimately dismiss concerns about naturality as easily as they are raised.

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Why don't set theorists have this attitude toward the consistency strengths?

Natural kinds vs. natural instances

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Distinction between natural kinds and natural instances.

Nonstandardness objection

Objection

In my incomparable statements, the consistency-strength differences are only revealed in ω-nonstandard models.

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Rebuttal

Consistency strength is *inherently* about ω-nonstandard models. If we only looked at ω-standard models, they would all agree about the consistency of any given theory whatsoever, and there would be no hierarchy to speak of.

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So we don't care about these differences.

Rebuttal

Consistency strength is *inherently* about ω-nonstandard models. If we only looked at ω-standard models, they would all agree about the consistency of any given theory whatsoever, and there would be no hierarchy to speak of.

To show $S < T$ in consistency strength requires a model of $\mathrm{Con}(S) + \neg\,\mathrm{Con}(T)$, which must be ω-nonstandard if T is consistent.

Confirmation bias error

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Confirmation-bias moral: it shouldn't be surprising to observe only linearity, if our tools cannot observe nonlinearity.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
Oxford University