The modal logic of forcing

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Introduction

I wish to investigate some of the general relations between the set-theoretic universe to its forcing extensions and grounds.

Guiding questions include:

Question

What are the properties of the set-theoretic universe in the context of its forcing extensions and grounds?

Question

What are the most general relations between forceability and truth?

A second-order enterprise

Those questions have a second-order aspect, but it turns out that first-order methods suffice for most of the analysis.

Several related projects:

- The modal logic of forcing.
  Upward oriented from a model to its forcing extensions.
  Identify the valid principles of forcing.
- Set-theoretic geology.
  Downward oriented from a model to its grounds.
  Investigate how the universe relates to its grounds.
- Generic multiverse setting.
  Mixed modal logic, both upward and downward.

This analysis engages pleasantly with various philosophical views on the nature of mathematical existence.

Many open questions remain.

Motivation

Question

What are the most general principles in set theory relating forcing and truth?

We are inspired by Solovay's analysis of provability $\mathrm{Pr}_T(\ulcorner\varphi\urcorner)$.

We aim to do for forceability what Solovay did for provability.

In both contexts, the questions and answers exhibit a modal nature and seem best formulated with modal logic.

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Forcing

Forcing (Cohen 1962) is a principal method of building models of set theory. It was used initially to prove the independence of Axiom of Choice and the Continuum Hypothesis.

Subsequent explosion of applications: enormous variety of models of set theory.

$$V \subseteq V[G]$$

The forcing extension $V[G]$ is built from the ground model V by adding a new ideal object G.

The extension $V[G]$ is closely related to the ground model V, but exhibits new truths in a way that can be carefully controlled.

Affinity of Forcing & Modal Logic

A ground model has access, via names and the forcing relation, to the objects and truths of the forcing extension. These are models of ZFC, so a mathematician might feel at home in any of them.

So there is a natural Kripke model lurking here.

- The *possible worlds* are the models of set theory.
- The *accessibility relation* relates a model M to its forcing extensions $M[G]$.

Many set theorists habitually operate within this Kripke model.

Modal operators

- A sentence φ is *possible* or *forceable*, written $\Diamond\varphi$, when it holds in a forcing extension.
- A sentence φ is *necessary*, written $\Box\varphi$, when it holds in all forcing extensions.

The modal assertions are expressible in set theory:

$$\begin{array}{ccc} \Diamond\varphi & \leftrightarrow & \exists\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi \\ \Box\varphi & \leftrightarrow & \forall\mathbb{P}\ \Vdash_{\mathbb{P}} \varphi \end{array}$$

While $\Diamond$ and $\Box$ are eliminable, we nevertheless retain them here, because we are interested in what principles these operators must obey.

Maximality Principle

The modal perspective arose in work on the the *maximality principle*, the scheme asserting that if φ can be forced in such a way that it remains true in all further extensions, then it is already true.

This can be expressed in the modal language as:

$$\Diamond\Box\varphi \to \varphi$$

Theorem (Stavi+Väänänen, indep. Hamkins)

If ZFC *is consistent, then so is ZFC + the maximality principle.*

Other modal principles of forcing?

Once one has the principle $\Diamond\Box\varphi \to \varphi$, then it is very natural to inquire:

- What other forcing axioms of this kind of modal form might we find?
- Which modal principles must be true under the forcing interpretation?
- In other words, what are the valid principles of forcing expressible in these modal terms?

Easy forcing validities

- K $\quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- Dual $\quad \Box\neg\varphi \leftrightarrow \neg\Diamond\varphi$
- S $\quad \Box\varphi \to \varphi$
- 4 $\quad \Box\varphi \to \Box\Box\varphi$
- .2 $\quad \Diamond\Box\varphi \to \Box\Diamond\varphi$

Easy Conclusion

Any S4.2 modal assertion is a valid principle of forcing.

A modal assertion $\varphi(p_0, \ldots, p_n)$ is a *valid principle of forcing* if $\varphi(\psi_0, \ldots, \psi_n)$ holds for any set theoretical ψ_i.

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What are the valid principles of forcing?

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What are the valid principles of forcing?

Beyond S4.2

5	$\Diamond\Box\varphi \to \varphi$
M	$\Box\Diamond\varphi \to \Diamond\Box\varphi$
W5	$\Diamond\Box\varphi \to (\varphi \to \Box\varphi)$
.3	$\Diamond\varphi \wedge \Diamond\psi \to (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond\varphi))$
Dm	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Diamond\Box\varphi \to \varphi)$
Grz	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$
Löb	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$
H	$\varphi \to \Box(\Diamond\varphi \to \varphi)$

It is a fun forcing exercise to show that these are invalid in some or all models of ZFC. Counterexamples built from such assertions as $V \neq L$, $\omega_1^L < \omega_1$, CH, or Boolean combinations of these.

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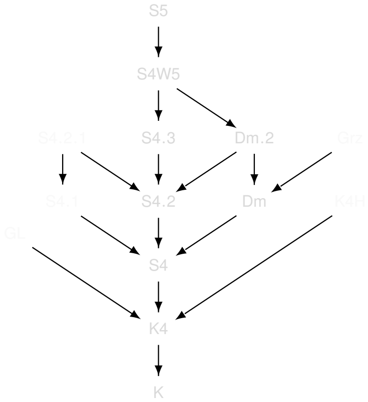
Some common modal theories

S5	=	S4 + 5
S4W5	=	S4 + W5
S4.3	=	S4 + .3
S4.2.1	=	S4 + .2 + M
S4.2	=	S4 + .2
S4.1	=	S4 + M
S4	=	K4 + S
Dm.2	=	S4.2 + Dm
Dm	=	S4 + Dm
Grz	=	K + Grz
GL	=	K4 + Löb
K4H	=	K4 + H
K4	=	K + 4
K	=	K + Dual

provably valid
provably invalid
consistently valid



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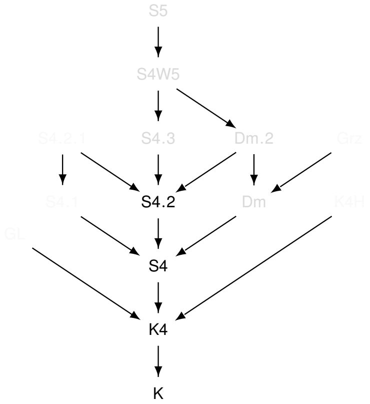

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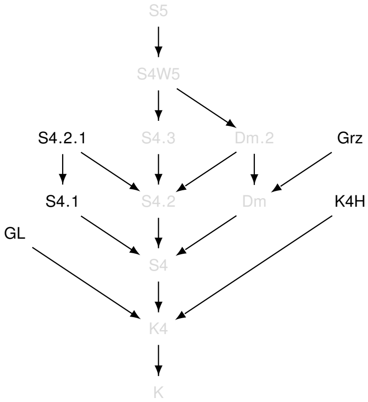

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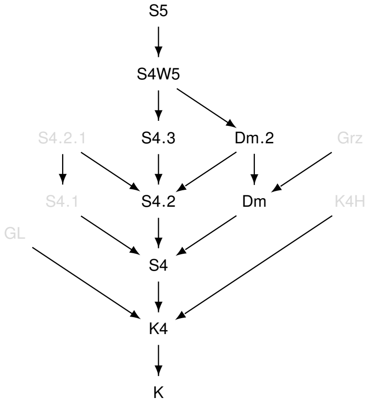

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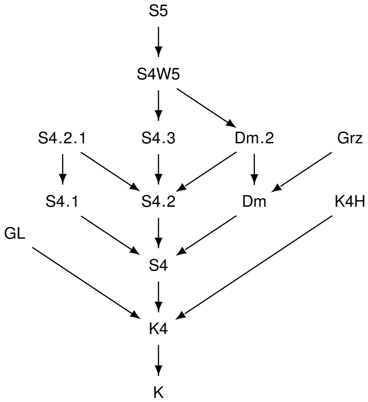

Valid principles of forcing

Theorem (Hamkins,Löwe)

If ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly S4.2*.*

We know S4.2 is valid. The difficult part is to show that nothing else is valid.

Given S4.2 $\nvdash \varphi$, we must provide ψ_i such that $\varphi(\psi_0, \ldots, \psi_n)$ fails in some model of set theory.

Control statements: Buttons and Switches

- A *switch* is a statement φ such that both φ and $\neg\varphi$ are necessarily possible.
- A *button* is a statement φ such that φ is (necessarily) possibly necessary.

Fact. Every statement in set theory is either a switch, a button or the negation of a button.

Theorem

If $V = L$, then there is an infinite independent family of buttons and switches.

Switches: $s_m = $ "GCH holds at $\aleph_{\omega+m}$"

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Kripke models and frames

The model theory and semantics of modal logic rests on the concept of a *Kripke model*: a collection of propositional *worlds*, with an underlying accessibility relation called the *frame*.

Partial pre-order implies S4. Directed pre-order implies S4.2

Fact

If S4.2 $\nvdash \varphi$, then φ fails in a Kripke model whose frame is a finite directed partial pre-order.

Improved Fact (Hamkins, Löwe)

If S4.2 $\nvdash \varphi$, then φ fails in a Kripke model whose frame is a finite pre-lattice.

Simulating Kripke models

Central Technical Lemma (Hamkins, Löwe)

If $W \models \mathrm{ZFC}$ has buttons and switches, then for any Kripke model M on a finite pre-lattice frame, any $w \in M$, there is $p_i \mapsto \psi_i$ so that for any φ:

$$(M, w) \models \varphi(p_1, \ldots, p_n) \quad \leftrightarrow \quad W \models \varphi(\psi_1, \ldots, \psi_n).$$

Consequently, if S4.2 $\nvdash \varphi$, then there is a substitution instance such that $W \models \neg\varphi(\psi_1, \ldots, \psi_n)$.

Main Theorem (Hamkins, Löwe)

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Validities in a model

Consider the valid principles of forcing over a fixed $W \models \mathrm{ZFC}$:

Theorem

$\mathrm{S4.2} \subseteq \mathrm{Force}^W \subseteq \mathrm{S5}$.

Both endpoints occur for various W.

Question

1. Can we have $\mathrm{S4.2} \subsetneq \mathrm{Force}^W \subsetneq \mathrm{S5}$?
2. If φ is valid for forcing over W, does it remain valid for forcing over all extensions of W? Equivalently, is Force^W normal?
3. Can a model of ZFC have an unpushed button, but not two independent buttons?

Between S4.2 $\subseteq \mathrm{Force}^W \subseteq$ S5

Theorem

Suppose $W \models \mathrm{ZFC}$.

1. *No unpushed buttons iff* $\mathrm{Force}^W = \mathrm{S5}$.
2. *Independent buttons+switches iff* $\mathrm{Force}^W = \mathrm{S4.2}$.
3. *Two semi-independent buttons* $\to$ W5 *invalid in* W.
4. *Two independent buttons* $\to$ .3 *invalid in* W.
5. *Independent button+switch* $\to$ Dm *invalid in* W.
6. *Long ratchets* $\to \mathrm{Force}^W \subseteq \mathrm{S4.3}$.

Forcing validities with parameters

With suitable parameters, one can always find independent buttons and switches. Consequently:

Theorem (Hamkins,Löwe)

The valid principles of forcing over any model of ZFC*, allowing arbitrary parameters from that model, are exactly those in the theory* S4.2*.*

Indeed, one only needs the parameters $\aleph_1, \aleph_2, \ldots$

Alternatively, one can also use as parameters countably many disjoint stationary subsets of ω_1.

With only real parameters, the validities can go beyond S4.2.

Entry of large cardinals

Theorem. The following are equiconsistent:

1. $S5(\mathbb{R})$ is valid.
2. $S4W5(\mathbb{R})$ is valid for forcing.
3. $Dm(\mathbb{R})$ is valid for forcing.
4. There is a stationary proper class of inaccessible cardinals.

Theorem

1. *(Welch,Woodin) If $S5(\mathbb{R})$ is valid in all forcing extensions (using $\mathbb{R}$ of extension), then $\mathrm{AD}^{L(\mathbb{R})}$.*
2. *(Woodin) If $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular, then it is consistent that $S5(\mathbb{R})$ is valid in all extensions.*

A class Γ

Consider any definable class Γ of forcing notions, such as c.c.c. forcing or cardinal-preserving forcing.

We have the corresponding notions of possibility and necessity:

- φ is Γ-possible, denoted $\Diamond_\Gamma \varphi$, if φ holds in some Γ-forcing extension.
- φ is Γ-necessary, denoted $\Box_\Gamma \varphi$, if φ holds in all Γ-forcing extensions.

To interpret iterated modalities $\Diamond_\Gamma \Box_\Gamma \varphi$, we re-interpret Γ in the extension.

Thus, we treat Γ *de dicto* rather than *de re*.

Restricted classes

Theorem (Hamkins, Löwe)

If ZFC *consistent, then provably valid principles of collapse-to-ω forcing and add-Cohen-reals forcing are exactly* S4.3

Theorem (Hamkins,Löwe)

Under MA*, the valid principles of* ccc *forcing include* S4.2*.*

Question

What are the provably valid principles of ccc forcing? proper forcing? class forcing? etc. etc.

.2 is not always valid. Class forcing and ccc surprisingly similar. Working conjecture: the modal logic of finite topless pre-Boolean algebras. Class forcing modalities not expressible.

Frame labeling

The main technical method of establishing upper bounds on the modal theories is provided by labeling the underlying frames for the logic.

A Γ*-labeling* of a frame F with a model W of set theory assigns nodes w in the frame to statements Φ_w of set theory, whose forceability relations mirror the structure of the frame:

- Every $W[G]$ satisfies exactly one Φ_w.
- Whenever $W[G] \models \Phi_w$, then $W[G]$ satisfies $\Diamond \Phi_u$ if and only if $w \leq_F u$.
- $W \models \Phi_{w_0}$, where w_0 is initial node of F.

These properties correspond to fulfilling the Jankov-Fine formula for the frame F.

Labelings limit modal logic

Technical Labeling Lemma

If $w \mapsto \Phi_w$ is a Γ-labeling of frame F for $W \models \mathrm{ZFC}$, then for any Kripke model M with frame F, there is an assignment $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, \ldots, p_k)$,

$$(M, w_0) \models \varphi(p_0, \ldots, p_k) \quad \text{iff} \quad W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

In particular, if φ fails at w_0 in M also fails in W under the Γ forcing interpretation.

Consequently, the modal logic of Γ forcing over W is contained in the modal logic of F validities.

Control statements

Many labelings can be constructed in modular fashion from various types of control statements.

- Switches $\Box\Diamond s \wedge \Box\Diamond\neg s$
- Buttons $\Box\Diamond\Box b$

  pushed $\Box b$, unpushed $\neg b \wedge \Diamond\Box b$, pure $\Box(b \to \Box b)$
- weak button $\Diamond\Box b$
- ratchets $r_1, r_2, \ldots r_n$ with

$$\begin{array}{l} \neg r_i \\ \Box(r_i \to \Box r_i) \\ \Box(r_{i+1} \to r_i) \\ \Box[\neg r_{i+1} \to \Diamond(r_i \wedge \neg r_{i+1})] \end{array}$$

- long ratchets r_α, continuous ratchets
- *Independent* controls

Controls → validities

Using the control statements, we can build frame labelings in a modular fashion, thereby providing the upper bounds on the forcing validities:

- Independent switches → validities contained in S5
- Long ratchet → validities contained in S4.3
- Buttons + switches → validities contained in S4.2
- Weak buttons + switches → validities contained in S4.tBA

Summary results

Theorem (Hamkins, Löwe)

If ZFC *is consistent, then*

1. *The provably valid principles of collapse-to-ω forcing are exactly* S4.3.
2. *The provably valid principles of adding-any-number-of Cohen-reals forcing are exactly* S4.3.
3. *The provably valid principles of countably-closed forcing are exactly* S4.2.
4. *Similarly for collapsing-to-κ, adding-to-κ and κ-closed, when κ is absolutely definable.*
5. *The provably valid principles of ω_1-preserving forcing are contained within* S4.tBA.

S4.3 as general upper bound

Corollary

The valid principles of forcing over L for each of the following forcing classes is contained within S4.3*.*

1. *c.c.c. forcing.*
2. *Proper forcing.*
3. *Semi-proper forcing.*
4. *Stationary-preserving forcing.*
5. ω_1*-preserving forcing.*
6. *Cardinal-preserving forcing.*
7. *Countably distributive forcing.*
8. κ*-distributive forcing, for a fixed absolutely definable cardinal* κ*.*

Collapse switches

An interesting question arose in connection with switches in the class of collapse-to-ω forcing.

Question (Hamkins, Löwe)

Can there be a model of ZFC, whose theory is fully invariant by collapse forcing $\mathrm{Coll}(\omega, \theta)$?

This would be a strong negative case of having no switches.

Welch and Mitchell independently showed that this situation has large cardinal strength: a measurable cardinal of high Mitchell degree.

Woodin provided an upper bound from determinacy.

Fuchs showed that if one allows arbitrary $\mathrm{Coll}(\gamma, \delta)$, then the answer is negative.

Reversing direction

Consider now the downward-oriented theory, where we look at possibility to grounds, rather than to forcing extensions.

Definition

A sentence φ is

ground possible, $\Diamond\varphi$, if it holds in a ground model
ground necessary, $\Box\varphi$, if it holds in all ground models.

Question

What is the downward modal logic of forcing?

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Mixed modal logic

What we really have here, of course, is a mixed modal logic, with two operators $\Diamond^{\wedge}$, $\Diamond^{\vee}$, allowing us to range over the generic multiverse.

What are the mixed validities? The answer theory may not have a name...

There is an affinity with temporal logic:

$$\varphi \longrightarrow (\overline{\Box}\,\Diamond^{\vee}\varphi) \wedge (\underline{\Box}\,\Diamond^{\wedge}\varphi)$$

Woodin's 3-step claim means that generic-multiverse possibility is equivalent to: $\Diamond^{\wedge}\,\Diamond^{\vee}\,\Diamond^{\wedge}\varphi$

Downward modal logic of forcing

It is easy to see that S4 is downward valid.

The validity of axiom $.2$, however, is open. $\Diamond \Box \varphi \to \Box \Diamond \varphi$

Definition

The *downward directed grounds* hypothesis DDG is the assertion that any two grounds of the universe have a common deeper ground. That is, the collection of grounds is downward directed.

Do any two models with a common forcing extension have a common ground?

The DDG holds in every model for which we are able to determine the answer.

Under the DDG, the theory S4.2 is downward valid.

Provable ground validities

Theorem (Hamkins, Löwe)

If ZFC *is consistent, then there is a model of* ZFC *whose valid downward principles of forcing are exactly* S4.2*.*

Proof Idea

Following set-theoretic geology, produce a model of ZFC having independent downward buttons and switches. Technical Lemma goes through, so we can simulate the relevant Kripke models.

Corollary

If ZFC is consistent and DDG is provable, then the ZFC-provably valid downward principles of forcing are exactly S4.2.

Without DDG, the validities are between S4 and S4.2.

Mixed modalities

We were able to attain various combinations of upward and downward modal forcing validities.

Theorem (Hamkins, Löwe)

If ZFC *is consistent, then there are models of* ZFC *whose valid principles of forcing are:*

1. *Simultaneously upward and downward* S4.2*.*
2. *Upward* S5 *and downward* S4.2*.*
3. *Upward* S4.2 *and downward* S5*.*

Maximality both up and down

Consider the joint maximality principle holding both up and down.

$$\overline{\Diamond}\,\overline{\Box}\,\varphi \longrightarrow \varphi$$

$$\underline{\Diamond}\,\underline{\Box}\,\varphi \longrightarrow \varphi$$

This is expressing S5 in both directions.

Theorem (Hamkins, Löwe)

The simultaneous up-and-down maximality principle is inconsistent.

Proof.

"All grounds have the same ω_1" is a downward button and an upward button negation. It cannot be pushed both ways. □

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The simultaneous up-and-down maximality principle is inconsistent.

Proof.

"All grounds have the same ω_1" is a downward button and an upward button negation. It cannot be pushed both ways. □

Future Directions

- Investigate range of validities of a fixed model $W \models \mathrm{ZFC}$.
- What is the modal logic of ccc forcing? proper forcing? etc.
- What is the modal logic of class forcing, or of arbitrary ZFC extensions?
- What is the mixed modal logic in the generic multiverse?

Thank you.

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