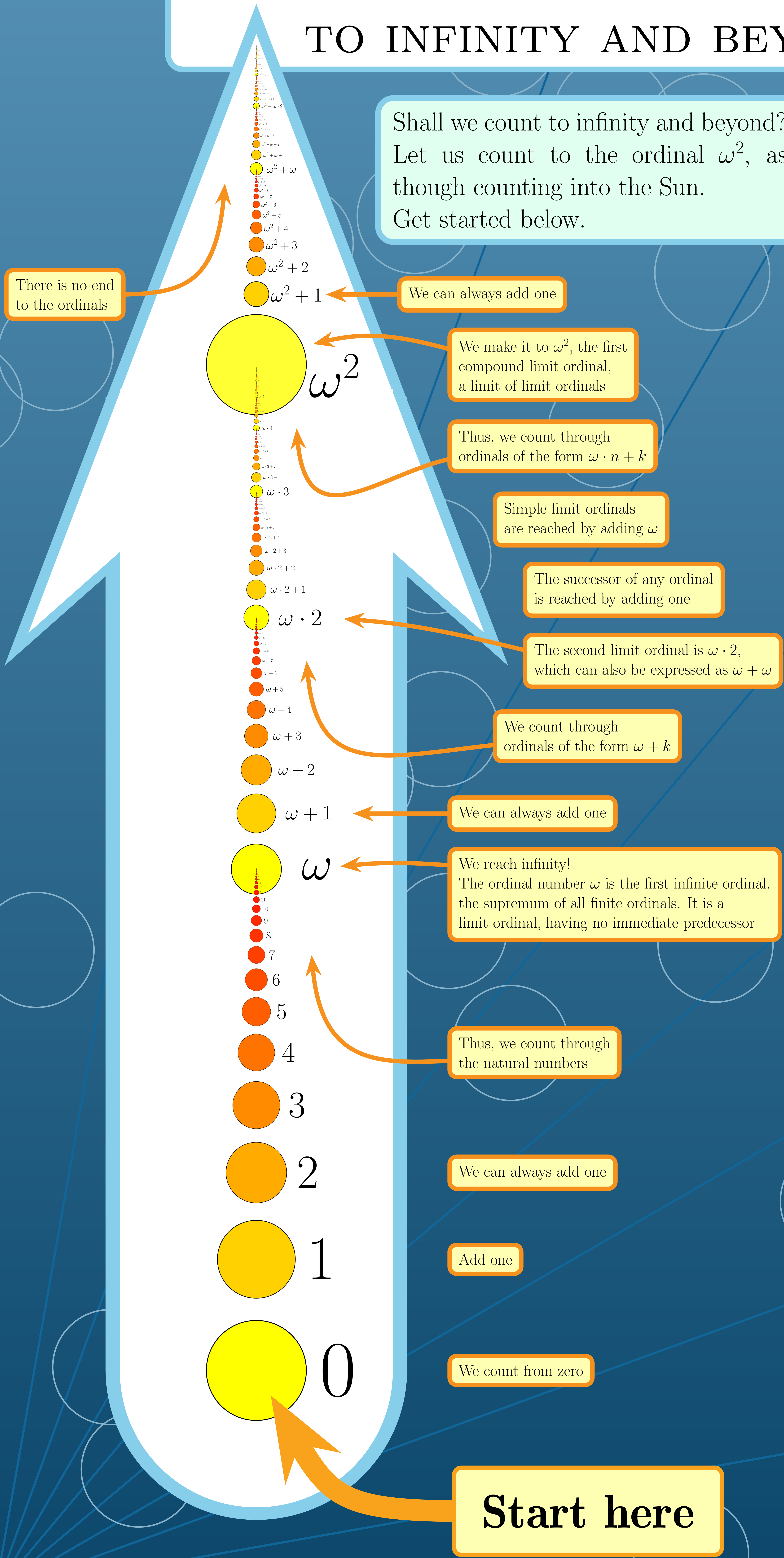
How to Count

to infinity and beyond

Shall we count to infinity and beyond? Let us count to the ordinal ω^2, as though counting into the Sun. Get started below.



Anyone can learn to count in the ordinals

Let us learn the names of the ordinals and how they proceed in order. Start with zero at the bottom and count through the natural numbers. The first infinite ordinal is ω (pronounced "omega"), which is a limit ordinal, because it has no immediate predecessor. Adding one—we can always add one—we form $\omega + 1$, $\omega + 2$, and so on, approaching $\omega \cdot 2$, the second limit ordinal, and ultimately the ordinals $\omega \cdot n + k$. The supremum is ω^2, the first compound limit ordinal, a limit of limit ordinals.

Just like counting to 100

When we count to one hundred, notice that within each decade—the teens, the twenties, the thirties—it is just like counting to ten. We thus count to ten altogether ten times. Similarly, when we count to ω^2, we encounter ω many eons, each of size ω. Counting to ω^2 is thus counting to ω altogether ω many times. And just as the numbers up to one hundred have two digits in base ten, similarly the ordinals up to ω^2 have the form $\omega \cdot n + k$, two digits in base ω.

No end to the ordinals

The ordinals look the same continuing forward from any point, as though starting anew. Beyond ω^2, we find ω^3, ω^ω, and the ordinal $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$, known as ϵ_0. Every set of ordinals has a supremum—the Church-Kleene ordinal ω_1^{CK} is the supremum of the computable ordinals; true ω_1 is the first uncountable ordinal. Continuing upward, we strive for the higher infinities of large cardinal set theory as the ordinal numbers pour without end through the transfinite hourglass.

Want to learn more?

The ordinals are a fundamental number system introduced by Georg Cantor in the 19th century and studied continuously since then in the subject known as set theory. Pick up a set theory book or take a set theory class to learn more!

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http://jdh.hamkins.org/counting-to-infinity-poster