Philosophy of Mathematics

University of Notre Dame Spring 2022

Philosophy of Mathematics
43906 01 (31349)
43906 02 (32481) - *Reserved for Glynn Honors Program*
Joel David Hamkins, O'Hara Professor of Philosophy and Mathematics
3:30-4:45 TR, DeBartolo Hall 301
Cross-listed with MATH 40920 01

This series of self-contained seminar lectures on the philosophy of mathematics is intended for students in philosophy and mathematics. The lectures will be organized loosely around mathematical themes, in such a way that brings various philosophical issues naturally to light.

Lecture and discussion will mainly follow my book:

> Joel David Hamkins, *Lectures on the Philosophy of Mathematics*, (MIT Press 2021).
> https://mitpress.mit.edu/books/lectures-philosophy-mathematics

There will also be a few supplemental readings interspersed through the course.

There are eight main topics, and each will be covered over a period of up to two weeks in the following manner:

- A few lecture sessions following the LPM text. We'll follow a flexible pace, rather than a strict schedule.
- A seminar discussion session based on the supplemental reading, for topics having such a reading.
- A class session going over the Questions for Further Thought from the LPM text.

Students are expected to engage fully with the Questions for Further Thought in the LPM text. For each topic, students should write brief essay-style solutions for about half or more of the questions at the end of each LPM chapter. Submit this work to my mailbox by early morning *the day before* the class going over the questions.

There will also be a final exam with questions similar to the Questions for Further Thought. The final grade will be based on the exam and on the submitted Questions for Further Thought work.

1 NUMBERS

Numbers are perhaps the essential mathematical idea, but what are numbers? There are many kinds of numbers---natural numbers, integers, rational numbers, real numbers, complex numbers, hyperreal numbers, surreal numbers, ordinal numbers, and more---and these number systems provide a fruitful background for classical arguments on incommensurability and transcendentality, while setting the stage for discussions of platonism, logicism, the nature of abstraction, the significance of categoricity, and structuralism.

Readings:

- LPM, chapter 1
- Benacerraf, Paul. 1965. *What numbers could not be*. The Philosophical Review 74 (1): 47–73. http://www.jstor.org/stable/2183530.

2 RIGOUR

Let us consider the problem of mathematical rigor in the development of the calculus. Informal continuity concepts and the use of infinitesimals ultimately gave way to the epsilon-delta limit concept, which secured a more rigorous foundation while also enlarging our conceptual vocabulary, enabling us to express more refined notions, such as uniform continuity, equicontinuity, and uniform convergence. Nonstandard analysis resurrected the infinitesimals on a more secure foundation, providing a parallel development of the subject. Meanwhile, increasing abstraction emerged in the function concept, which we shall illustrate with the Devil's staircase, space-filling curves, and the Conway base 13 function. Finally, does the indispensability of mathematics for science ground mathematical truth? Fictionalism puts this in question.

Readings:

- LPM, chapter 2

3 INFINITY

We shall follow the allegory of Hilbert's hotel and the paradox of Galileo to the equinumerosity relation and the notion of countability. Cantor's diagonal arguments, meanwhile, reveal uncountability and a vast hierarchy of different orders of infinity; some arguments give rise to the distinction between constructive and nonconstructive proof. Zeno's paradox highlights classical ideas on potential versus actual infinity. Furthermore, we shall count into the transfinite ordinals.

Readings:

- LPM, chapter 3

4 GEOMETRY

Classical Euclidean geometry is the archetype of a mathematical deductive process. Yet the impossibility of certain constructions by straightedge and compass, such as doubling the cube, trisecting the angle, or squaring the circle, hints at geometric realms beyond Euclid. The rise of non-Euclidean geometry, especially in light of scientific theories and observations suggesting that physical reality is not Euclidean, challenges previous accounts of what geometry is about. New formalizations, such as those of David Hilbert and Alfred Tarski, replace the old axiomatizations, augmenting and correcting Euclid with axioms on completeness and betweenness. Ultimately, Tarski's decision procedure points to a tantalizing possibility of automation in geometrical reasoning.

Readings:

- LPM, chapter 4

5 PROOF

What is proof? What is the relation between proof and truth? Is every mathematical truth true for a reason? After clarifying the distinction between syntax and semantics and discussing various views on the nature of proof, including proof-as-dialogue, we shall consider the nature of formal proof. We shall highlight the importance of soundness, completeness, and verifiability in any formal proof system, outlining the central ideas used in proving the completeness theorem. The compactness property distills the finiteness of proofs into an independent, purely semantic consequence. Computer-verified proof promises increasing significance; its role is well illustrated by the history of the four-color theorem. Nonclassical logics, such as intuitionistic logic, arise naturally from formal systems by weakening the logical rules.

Readings:

- LPM, chapter 5

6 COMPUTABILITY

What is computability? Kurt Gödel defined a robust class of computable functions, the primitive recursive functions, and yet he gave reasons to despair of a fully satisfactory answer. Nevertheless, Alan Turing's machine concept of computability, growing out of a careful philosophical analysis of the nature of human computability, proved robust and laid a foundation for the contemporary computer era; the widely accepted Church-Turing thesis asserts that Turing had the right notion. The distinction between computable decidability and computable enumerability, highlighted by the undecidability of the halting problem, shows that not all mathematical problems can be solved by machine, and a vast hierarchy looms in the Turing degrees, an infinitary information theory. Complexity theory refocuses the subject on the realm of feasible computation, with the still-unsolved P versus NP problem standing in the background of nearly every serious issue in theoretical computer science.

Readings:

- LPM, chapter 6
- Turing, A. M. 1936. *On Computable Numbers*, with an Application to the Entscheidungsproblem. Proceedings of the London Mathematical Society 42 (3): 230–265. https://doi.org/10.1112/plms/s2-42.1.230.
- Aaronson, Scott. 2006. *Reasons to believe*. Shtetl-Optimized, the blog of Scott Aaronson. https://www.scottaaronson.com/blog/?p=122.

7 INCOMPLETENESS

David Hilbert sought to secure the consistency of higher mathematics by finitary reasoning about the formalism underlying it, but his program was dashed by Gödel's incompleteness theorems, which show that no consistent formal system can prove even its own consistency, let alone the consistency of a higher system. We shall describe several proofs of the first incompleteness theorem, via the halting problem, self-reference, and definability, showing senses in which we cannot complete mathematics. After this, we shall discuss the second incompleteness theorem, the Rosser variation, and Tarski's theorem on the nondefinability of truth. Ultimately, one is led to the inherent hierarchy of consistency strength rising above every foundational mathematical theory.

Readings:

- LPM, chapter 7

8 SET THEORY

We shall discuss the emergence of set theory as a foundation of mathematics. Cantor founded the subject with key set-theoretic insights, but Frege's formal theory was naive, refuted by the Russell paradox. Zermelo's set theory, in contrast, grew ultimately into the successful contemporary theory, founded upon a cumulative conception of the set-theoretic universe. Set theory was simultaneously a new mathematical subject, with its own motivating questions and tools, but it also was a new foundational theory with a capacity to represent essentially arbitrary abstract mathematical structure. Sophisticated technical developments, including in particular, the forcing method and discoveries in the large cardinal hierarchy, led to a necessary engagement with deep philosophical concerns, such as the criteria by which one adopts new mathematical axioms and set-theoretic pluralism.

Readings:

- LPM, chapter 8
- Maddy, Penelope. 2017. *Set-theoretic foundations*. In Foundations of Mathematics, eds. Andrés Eduardo Caicedo, James Cummings, Peter Koellner, and Paul B. Larson. Vol. 690 of Contemporary Mathematics Series, 289–322. American Mathematical Society. https://doi.org/10.1090/conm/690, preprint available here: http://philsci-archive.pitt.edu/13027/.