Set-theoretic mereology as a foundation of mathematics?

Joel David Hamkins
O'Hara Professor of Philosophy and Mathematics
University of Notre Dame

Guest Chair Professor
Peking University

Workshop on Mereology
Shandong University, Jinan, China
22 June 2026

Joint work

This talk contains results of joint work with Makoto Kikuchi.

[HK17] J. D. Hamkins and M. Kikuchi, The inclusion relations of the countable models of set theory are all isomorphic, Mathematics ArXiv, 2017.

[HK16] J. D. Hamkins and M. Kikuchi, Set-theoretic mereology, Logic and Logical Philosophy, special issue “Mereology and beyond, part II”, vol. 25, iss. 3, pp. 1-24, 2016.

A review of mereology

Mereology is the study of the relation of part to whole.

The focus of study is on the parthood relation:

$p \sqsubseteq q$ expresses that p is a *part* of q.

Mereology—a brief history

- Classical roots in the Presocratics

Mereology—a brief history

- Classical roots in the Presocratics
- Continues through Plato, Aristotle...
- and medieval ontologists and scholastic philosophers

Mereology—a brief history

- Classical roots in the Presocratics
- Continues through Plato, Aristotle...
- and medieval ontologists and scholastic philosophers
- Formal theory develops with Brentano and Husserl (1901)
- Especially with Leśniewski's *Foundations of the General Theory of Sets* (1916) and his *Foundations of Mathematics* (1927–1931).

Mereology—a brief history

- Classical roots in the Presocratics
- Continues through Plato, Aristotle...
- and medieval ontologists and scholastic philosophers
- Formal theory develops with Brentano and Husserl (1901)
- Especially with Leśniewski's *Foundations of the General Theory of Sets* (1916) and his *Foundations of Mathematics* (1927–1931).
- David Lewis, *Parts of Classes* (1991).
- Active area of current research.

A rich ontology for the mereological conception

One mereological notion is that of a *fusion* or *sum:* the whole composed of some given parts. The fusion of all cats is that large, scattered chunk of cat-stuff which is composed of all the cats there are, and nothing else. It has all cats as parts. There are other things that have all cats as parts. But the cat-fusion is the least such thing: it is included as a part in any other one.

It does have other parts too: all cat-parts are parts of it, for instance cat-whiskers, cat-quarks. For parthood is transitive: whatever is part of a cat is thereby part of a part of the cat-fusion, and so must itself be part of the cat-fusion.

The cat-fusion has still other parts. We count it as a part of itself: an *improper* part, a part identical to the whole. But is also has plenty of *proper parts*—parts not identical to the whole—besides the cats and cat-parts already mentioned. Lesser fusions of cats, for instance the fusion of my two cats Magpie and Possum, are proper parts of the grand fusion of all cats. Fusions of cat-parts are parts of it too, for instance the fusion of Possum's paws plus Magpie's whiskers, or the fusion of all cat-tails wherever they be. Fusions of several cats plus several cat-parts are parts of it. And yet the cat-fusion is made of nothing but cats, in this sense: it has no part that is entirely distinct from each and every cat. Rather, every part of it overlaps some cat.

–David Lewis, Parts of Classes [Lew91]

Mereology vs. set theory

Mereology is often contrasted with set theory and its membership relation, the relation of element to set.

The two subjects appeared as formal theories at about the same time.

Mereology vs. set theory

Whereas mereology focuses on the *parthood* relation.

$$p \sqsubseteq q \qquad \text{object } p \text{ is a part of object } q.$$

Mereology vs. set theory

Whereas mereology focuses on the *parthood* relation.

$p \sqsubseteq q$ object p is a part of object q.

Set theory focuses on the *element of* relation:

$p \in q$ object p is an element of set q.

The $\in$ relation is not fundamentally mereological.

Examples of a few mereological axioms

Various fundamental mereological principles.

- Reflexivity: $p \sqsubseteq p$
- Transitivity: $p \sqsubseteq q \sqsubseteq r \implies p \sqsubseteq r$
- Antisymmetry: $p \sqsubseteq q \sqsubseteq p \implies p = q$

Examples of a few mereological axioms

Various fundamental mereological principles.

- Reflexivity: $p \sqsubseteq p$
- Transitivity: $p \sqsubseteq q \sqsubseteq r \implies p \sqsubseteq r$
- Antisymmetry: $p \sqsubseteq q \sqsubseteq p \implies p = q$
- Mereological sum: $\forall p, q \quad \exists\ p \sqcup q$
- Mereological difference: $\forall p, q \quad \exists\ p \setminus q$

Examples of a few mereological axioms

Various fundamental mereological principles.

- Reflexivity: $p \sqsubseteq p$
- Transitivity: $p \sqsubseteq q \sqsubseteq r \implies p \sqsubseteq r$
- Antisymmetry: $p \sqsubseteq q \sqsubseteq p \implies p = q$
- Mereological sum: $\forall p, q \quad \exists\, p \sqcup q$
- Mereological difference: $\forall p, q \quad \exists\, p \setminus q$
- Atomicity: every p is the fusion of atoms

Foundation of mathematics

Set theory has found robust success as a foundation of mathematics.

Set theory has the capacity to express and represent essentially arbitrary mathematical structure.

Foundation of mathematics

Set theory has found robust success as a foundation of mathematics.

Set theory has the capacity to express and represent essentially arbitrary mathematical structure.

In light of this power, Hilbert proclaimed,

> No-one shall cast us from the paradise that Cantor has created for us. [Hil26]

Faithful representations in set theory

Yiannis Moschovakis summarizes the attitude:

> ...we will discover within the universe of sets *faithful representations* of all the mathematical objects we need, and we will study set theory on the bases of the lean axiomatic system of Zermelo **as if all mathematical objects were sets**. The delicate problem in specific cases is to formulate precisely the correct definition of a “faithful representation” and to prove that one such exists. *[Mos06, p. 34]*

Thus, set theory becomes a grand unified theory of mathematics.

Main question

Both set theory and mereology offer a fundamental relation of sweeping abstraction and generality.

Yet, while set theory has found success in the foundation of mathematics, mereology has been strangely absent.

Main Question and Theme

Why has mereology not succeeded as a foundation of mathematics?

Set-theoretic mereology

I propose to to analyze the question in the context of *set-theoretic* mereology, the study of the set-theoretic inclusion relation:

$$p \subseteq q$$

This is the natural mereological relation arising in set theory.

This relation is the focus of David Lewis's mereological conception in Parts and Classes [Lew91].

Main questions

Question

Can set-theoretic mereology serve as a foundation of mathematics?

That is, can we found mathematics upon the inclusion relation $\subseteq$ rather than the element-of relation $\in$?

Can the inclusion relation $\subseteq$ provide faithful representations of arbitrary mathematical structure?

At bottom: Can we get by with merely $\subseteq$ in place of $\in$ in the foundations of mathematics?

The project

Working in the universe of set theory $\langle V, \in \rangle$, consider the *mereological reduct* structure, $\langle V, \subseteq \rangle$.

Question

How well does $\langle V, \subseteq \rangle$ serve as a foundation of mathematics?

The study of the structure $\langle V, \subseteq \rangle$ is *set-theoretic mereology*.

Pure mereology vs. augmented mereology

If we augment mereology with the singleton operator

$$a \mapsto \{a\}$$

the resulting theory becomes inter-definable with $\in$-set theory.

We can define

$$x \in y \quad \leftrightarrow \quad \{x\} \subseteq y.$$

Pure mereology vs. augmented mereology

If we augment mereology with the singleton operator

$$a \mapsto \{a\}$$

the resulting theory becomes inter-definable with $\in$-set theory.

We can define

$$x \in y \quad \leftrightarrow \quad \{x\} \subseteq y.$$

Consequently, the two structures are definitionally equivalent:

$$\langle V, \in \rangle \qquad \text{and} \qquad \langle V, \subseteq, \{\cdot\} \rangle.$$

In my view, mereology with singletons IS set theory.

Different $\in$, same $\subseteq$?

Question (Kikuchi)

Can there be two models of set theory with different membership relations $\in$, but the same inclusion relation $\subseteq$?

Kikuchi asks for models of set theory $\langle V, \in\rangle$ and $\langle V, \in^*\rangle$ on the same domain of sets, with distinct membership relations $\in$, $\in^*$, but with the identical inclusion relations $\subseteq$ and $\subseteq^*$.

Different $\in$, same $\subseteq$?

Question (Kikuchi)

Can there be two models of set theory with different membership relations $\in$, but the same inclusion relation $\subseteq$?

Kikuchi asks for models of set theory $\langle V, \in\rangle$ and $\langle V, \in^*\rangle$ on the same domain of sets, with distinct membership relations $\in$, $\in^*$, but with the identical inclusion relations $\subseteq$ and $\subseteq^*$.

The answer is yes.

Different $\in$, same $\subseteq$

Theorem (Hamkins, Kikuchi [HK16])

Every universe of set theory $\langle V, \in\rangle$ admits an alternative membership relation $\in^*$, for which the two inclusion relations are identical.

$$\subseteq^* \quad = \quad \subseteq$$

In fact, $\langle V, \in\rangle$ and $\langle V, \in^*\rangle$ will be isomorphic. And $\in^*$ will be definable.

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,and let

$$\tau : u \quad \mapsto \quad \theta \,"\, u = \{\, \theta(a) \mid a \in u \,\}.$$

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,and let

$$\tau : u \quad \mapsto \quad \theta \,\text{"}\, u = \{\, \theta(a) \mid a \in u \,\}.$$

Since θ is bijective and definable, this also is bijective (and nontrivial).

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,and let

$$\tau : u \quad \mapsto \quad \theta \text{ " } u = \{\, \theta(a) \mid a \in u \,\}.$$

Since θ is bijective and definable, this also is bijective (and nontrivial).

Notice that τ is an $\subseteq$-automorphism, since

$$u \subseteq v \quad \leftrightarrow \quad \theta \text{ " } u \subseteq \theta \text{ " } v \quad \leftrightarrow \quad \tau(u) \subseteq \tau(v).$$

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,and let

$$\tau : u \quad \mapsto \quad \theta \,"\, u = \{\, \theta(a) \mid a \in u \,\}.$$

Since θ is bijective and definable, this also is bijective (and nontrivial).

Notice that τ is an $\subseteq$-automorphism, since

$$u \subseteq v \quad \leftrightarrow \quad \theta \,"\, u \subseteq \theta \,"\, v \quad \leftrightarrow \quad \tau(u) \subseteq \tau(v).$$

Define $a \in^* b \leftrightarrow \tau(a) \in \tau(b)$. So τ is an isomorphism of $\langle V, \in^* \rangle$ with $\langle V, \in \rangle$.

Proof: Different $\in$, same $\subseteq$

Let $\theta : V \to V$ be any definable non-identity permutation,and let

$$\tau : u \quad\mapsto\quad \theta\, "\, u = \{\, \theta(a) \mid a \in u \,\}.$$

Since θ is bijective and definable, this also is bijective (and nontrivial).

Notice that τ is an $\subseteq$-automorphism, since

$$u \subseteq v \quad\leftrightarrow\quad \theta\, "\, u \subseteq \theta\, "\, v \quad\leftrightarrow\quad \tau(u) \subseteq \tau(v).$$

Define $a \in^* b \leftrightarrow \tau(a) \in \tau(b)$. So τ is an isomorphism of $\langle V, \in^*\rangle$ with $\langle V, \in\rangle$.

Yet, $u \subseteq^* v \leftrightarrow \forall a\, (a \in^* u \to a \in^* v)$, which holds iff $\forall a (\tau(a) \in \tau(u) \to \tau(a) \in \tau(v))$; but since τ is surjective, this holds iff $\tau(u) \subseteq \tau(v)$, which holds iff $u \subseteq v$.

So $\in$ and $\in^*$ are distinct, but $\subseteq$ and $\subseteq^*$ are identical, as desired. $\square$

$\in$ is not definable from $\subseteq$

Corollary

One cannot define $\in$ from $\subseteq$ in any model of set theory, even allowing parameters in the definition.

Proof.

If we could define $\in$ from $\subseteq$, then the same definition would have to define $\in^*$, which is different. □

τ is an isomorphism of $\langle V, \in^*, \subseteq\rangle$ with $\langle V, \in, \subseteq\rangle$.

Can $\subseteq$ provide a foundation of mathematics?

If we had been able to define $\in$ from $\subseteq$, then certainly $\subseteq$ would have been robust enough to serve as a foundation of mathematics.

But it isn't definable.

This is telling for the main question, but by itself, this doesn't show that that $\subseteq$-mereology cannot provide a foundation, since perhaps it can interpret structure in some other way.

We seek now to show that $\subseteq$-mereology cannot serve as a foundation of mathematics.

Good news/bad news

Good news/bad news

Theorem (Hamkins,Kikuchi)

Set-theoretic mereology is a decidable theory.

Good news/bad news

Theorem (Hamkins,Kikuchi)

Set-theoretic mereology is a decidable theory.

There is a computational procedure that will decide the truth of any statement in the structure $\langle V, \subseteq \rangle$.

Good news/bad news

Theorem (Hamkins,Kikuchi)

Set-theoretic mereology is a decidable theory.

There is a computational procedure that will decide the truth of any statement in the structure $\langle V, \subseteq \rangle$.

This may sound at first like good news.

Good news/bad news

Theorem (Hamkins,Kikuchi)

Set-theoretic mereology is a decidable theory.

There is a computational procedure that will decide the truth of any statement in the structure $\langle V, \subseteq \rangle$.

This may sound at first like good news.

But ultimately, our view is that for any attempt to use set-theoretic mereology as a foundation of mathematics, this is devastating.

What set-theoretic mereology is

Consider the set-theoretic universe $\langle V, \subseteq\rangle$.

Theorem

Set-theoretic mereology, considered as the theory of $\langle V, \subseteq\rangle$, is precisely the theory of an atomic unbounded relatively complemented distributive lattice, and furthermore, this theory is finitely axiomatizable, complete and decidable.

What set-theoretic mereology is

Consider the set-theoretic universe $\langle V, \subseteq \rangle$.

Theorem

Set-theoretic mereology, considered as the theory of $\langle V, \subseteq \rangle$, is precisely the theory of an atomic unbounded relatively complemented distributive lattice, and furthermore, this theory is finitely axiomatizable, complete and decidable.

This theorem will proceed by quantifier-elimination in the style of Tarski's classification of Boolean algebras and Eršov's extension to relatively complemented distributive lattices.

What set-theoretic mereology is

The first part is easy:

Theorem

If $\langle W, \in^W \rangle$ is a model of set theory, then $\langle W, \subseteq \rangle$ is an atomic unbounded relatively complemented distributive lattice.

What set-theoretic mereology is

The first part is easy:

Theorem

If $\langle W, \in^W \rangle$ is a model of set theory, then $\langle W, \subseteq \rangle$ is an atomic unbounded relatively complemented distributive lattice.

This is the nature of $\subseteq$ in any model of set theory.

Quantifier elimination

The quantifier-elimination part is more difficult.

Theorem

The theory of atomic unbounded relatively complemented distributive lattices admits elimination of quantifiers in the language with inclusion $\subseteq$, intersection $x \cap y$, union $x \cup y$, relative complement $x - y$ and the unary size relations $|x| = n$ and $|x| \geq n$, for each natural number n.

Quantifier elimination

The quantifier-elimination part is more difficult.

Theorem

The theory of atomic unbounded relatively complemented distributive lattices admits elimination of quantifiers in the language with inclusion $\subseteq$, intersection $x \cap y$, union $x \cup y$, relative complement $x - y$ and the unary size relations $|x| = n$ and $|x| \geq n$, for each natural number n.

What this means is that every assertion in the language of $\subseteq$ is equivalent to a quantifier-free assertion in the language $\subseteq, \cap, \cup, x - y$, and $|x| = n$, $|x| \geq n$.

The proof proceeds by induction on formulas, eliminating $\exists x$.

Some ideas from the proof

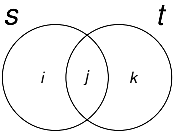
By induction on formulas. Suffices to eliminate quantifier from $\exists x\, \varphi(x, y_0, \ldots, y_n)$, where φ is q-free in expanded language.

- Can eliminate equality via $x = y \leftrightarrow x \subseteq y \subseteq x$.
- Eliminate negation via

$$\neg(|t| \geq n) \leftrightarrow \bigvee_{k<n} |t| = k \quad \text{and}$$

$$\neg(|t| = n) \leftrightarrow (|t| \geq n+1) \vee \bigvee_{k<n} |t| = k.$$

More ideas from the proof



Reduce size assertions to cell terms in Venn diagram:

$$|s \cup t| = n \;\leftrightarrow \bigvee_{i+j+k=n} (|s| = i + j) \wedge (|s \cap t| = j) \wedge (|t| = j + k)$$

$$|s \cup t| \;\geq n \leftrightarrow \bigvee_{i+j+k=n} (|s - t| \geq i) \wedge (|s \cap t| \geq j) \wedge (|t - s| \geq k).$$

Further ideas from the proof

Ultimately, one reduces to conjunction of positive assertions about cells in the Venn diagram, for which $\exists x$ is eliminable.

Further ideas from the proof

Ultimately, one reduces to conjunction of positive assertions about cells in the Venn diagram, for which $\exists x$ is eliminable.

For example,

$$\exists x\,(|x\cap c|\geq 3)\wedge(|x\cap c|\geq 7)\wedge(|c-x|=2)$$

is equivalent to

$$|c|\geq 9.$$

Set-theoretic mereology is complete

It follows from the quantifier-elimination result that set-theoretic mereology is a complete theory, because every sentence is equivalent to a quantifier-free sentence in that language, and these sentences are all trivially decided by the theory.

Set-theoretic mereology is complete

It follows from the quantifier-elimination result that set-theoretic mereology is a complete theory, because every sentence is equivalent to a quantifier-free sentence in that language, and these sentences are all trivially decided by the theory.

So set-theoretic mereology is finitely axiomatizable and complete.

Mereology is decidable

Once one knows that set-theoretic mereology is finitely axiomatizable and complete, then it follows that the theory is decidable.

Mereology is decidable

Once one knows that set-theoretic mereology is finitely axiomatizable and complete, then it follows that the theory is decidable.

Given any question, simply search for a proof of the statement or its negation.

Mereology for finite sets is same as for all sets

Corollary

In set-theoretic mereology, the hereditarily finite sets form an elementary substructure of the full universe

$$\langle \mathrm{HF}, \subseteq \rangle \prec \langle V, \subseteq \rangle$$

Mereology for finite sets is same as for all sets

Corollary

In set-theoretic mereology, the hereditarily finite sets form an elementary substructure of the full universe

$$\langle \mathrm{HF}, \subseteq \rangle \prec \langle V, \subseteq \rangle$$

Proof.

Both are atomic unbounded relatively complemented distributive lattices; so both support the quantifier-elimination procedure. And they agree on quantifier-free truth. □

Conclusion

We may conclude that set-theoretic mereology is weak.

Conclusion

We may conclude that set-theoretic mereology is weak.

Because $\langle \mathrm{HF}, \subseteq \rangle$ has exactly the same mereological truths as $\langle V, \subseteq \rangle$, it follows that set-theoretic mereology is unable to express the existence of infinite sets; or indeed any other truths of the transfinite.

Pure mereology seems to be missing out on fundamental aspects of the set-theoretic universe.

Why mereology will not provide a foundation

Regardless of whether set-theoretic mereology interprets the set-theoretic universe, we argue furthermore that ultimately, it cannot provide a foundation of mathematics.

- Set-theoretic mereology is a decidable theory.

Why mereology will not provide a foundation

Regardless of whether set-theoretic mereology interprets the set-theoretic universe, we argue furthermore that ultimately, it cannot provide a foundation of mathematics.

- Set-theoretic mereology is a decidable theory.
- No undecidable theory is faithfully represented in a decidable theory.

Why mereology will not provide a foundation

Regardless of whether set-theoretic mereology interprets the set-theoretic universe, we argue furthermore that ultimately, it cannot provide a foundation of mathematics.

- Set-theoretic mereology is a decidable theory.
- No undecidable theory is faithfully represented in a decidable theory.
- Arithmetic and many other fundamental mathematical theories are undecidable.

Why mereology will not provide a foundation

Regardless of whether set-theoretic mereology interprets the set-theoretic universe, we argue furthermore that ultimately, it cannot provide a foundation of mathematics.

- Set-theoretic mereology is a decidable theory.
- No undecidable theory is faithfully represented in a decidable theory.
- Arithmetic and many other fundamental mathematical theories are undecidable.
- Therefore, set-theoretic mereology cannot serve as a foundation of mathematics.

Extending to other pure mereologies

A similar argument applies to other forms of pure mereology, for example, if one gives up atomicity.

Extending to other pure mereologies

A similar argument applies to other forms of pure mereology, for example, if one gives up atomicity.

The reason is that the Tarski/Eršov's analysis applies to any Boolean algebra or relatively complemented distributive lattice.

Extending to other pure mereologies

A similar argument applies to other forms of pure mereology, for example, if one gives up atomicity.

The reason is that the Tarski/Eršov's analysis applies to any Boolean algebra or relatively complemented distributive lattice.

These also are decidable theories, and so they cannot provide a foundation of mathematics.

Answering the main question

In short, I claim that the answer to the main question consists of the observation:

Answering the main question

In short, I claim that the answer to the main question consists of the observation:

Main philosophical conclusion

Mereology has not provided a foundation of mathematics—and it cannot provide a foundation of mathematics—because it is a decidable theory.

Answering the main question

In short, I claim that the answer to the main question consists of the observation:

Main philosophical conclusion

Mereology has not provided a foundation of mathematics—and it cannot provide a foundation of mathematics—because it is a decidable theory.

In particular, in my view the issue of decidability should become a core part of the discussion of mereology as a foundational theory.

Which models arise?

Let me turn now to investigate the model theory of mereology.

Which models arise?

Let me turn now to investigate the model theory of mereology.

We have seen that every model of set theory $\langle M, \in^M \rangle$ gives rise to its corresponding model of mereology $\langle M, \subseteq^M \rangle$.

Which models arise?

Let me turn now to investigate the model theory of mereology.

We have seen that every model of set theory $\langle M, \in^M \rangle$ gives rise to its corresponding model of mereology $\langle M, \subseteq^M \rangle$.

Question

Which models of the theory of set-theoretic mereology arise as such mereological reducts $\subseteq^M$ of a model of set theory?

Mereology has only one countable model

The surprising answer is that there is only one countable model of set-theoretic mereology.

They are all isomorphic.

Mereology has only one countable model

The surprising answer is that there is only one countable model of set-theoretic mereology.

They are all isomorphic.

Theorem

All countable models of set theory $\langle M, \in^M \rangle \models$ ZFC have the same mereological reduct $\langle M, \subseteq^M \rangle$ up to isomorphism.

The result applies to much weaker set theories than ZFC, but there is an interesting use of AC.

Saturation

The categoricity theorem is a consequence of the following:

Theorem

The mereological reducts $\langle M, \subseteq^M \rangle$ of countable models of set theory are precisely the countable saturated models of set-theoretic mereology.

Saturation

The categoricity theorem is a consequence of the following:

Theorem

The mereological reducts $\langle M, \subseteq^M \rangle$ of countable models of set theory are precisely the countable saturated models of set-theoretic mereology.

All saturated models of a complete theory are isomorphic by the back-and-forth method.

Affinity with arithmetic

Analogous result in arithmetic:

Theorem (Lipshitz/Nadel 1978)

Every nonstandard model of arithmetic $\langle M, \cdot, +, 0, 1, < \rangle$ *has a computably saturated reduct to addition* $\langle M, + \rangle$.

The model $\langle M, + \rangle$ is determined by the standard system of M.

Expressive power of types in mereology

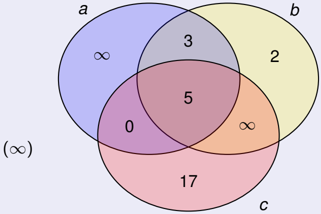
The theorem is proved by understanding the power of types in set-theoretic mereology.

Expressive power of types in mereology

The theorem is proved by understanding the power of types in set-theoretic mereology.

Lemma

If $p(a_1, \ldots, a_n)$ is a complete n-type in the language of set-theoretic mereology, then $p(a_1, \ldots, a_n)$ is equivalent over the theory of set-theoretic mereology to the assertions stating for each cell in the Venn diagram of the variables that it has some specific finite size or that it is infinite.



$$|a - (b \cup c)| = \infty$$
$$|(a \cap b) - c| = 3$$
$$|b - (a \cup c)| = 2$$
$$|(a \cap c) - b| = 0$$
$$|a \cap b \cap c| = 5$$
$$|(b \cap c) - a| = \infty$$
$$|c - (a \cup b)| = 17$$

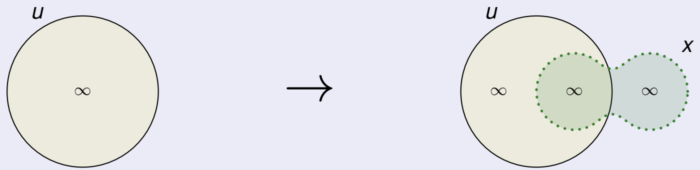
Saturated models of set-theoretic mereology

Theorem

A model of set-theoretic mereology $\langle M, \sqsubseteq\rangle$ is $\aleph_0$-saturated if and only if

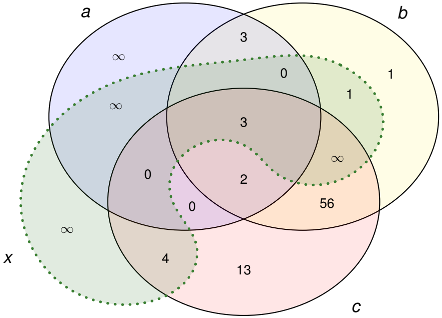
1. *every infinite element of M is the disjoint union of two infinite elements, and*
2. *for every element $a \in M$, there is an infinite element $u \in M$ disjoint from a.*

Equivalently, there are infinite elements and for every infinite element u there is an element x for which $u - x$, $u \cap x$ and $x - u$ are each infinite.



Proof idea

The type asserts of x and the parameters that a certain pattern of sizes is realized in the Venn diagram.



A complete type $p(x, a, b, c)$ makes assertions about how x splits the cells in the Venn diagram of a, b and c and how much of x is outside $a \cup b \cup c$.

The hypothesis ensures that we can assemble such an x. So the model is saturated. □

What mereology knows

Because there is only one countable model of set-theoretic mereology, the inclusion relation $\subseteq$ carries essentially NO information about the ambient set-theoretic universe $\in$.

Conclusion

Set-theoretic mereology doesn't know any set theory.

What mereology knows

Because there is only one countable model of set-theoretic mereology, the inclusion relation $\subseteq$ carries essentially NO information about the ambient set-theoretic universe $\in$.

Conclusion

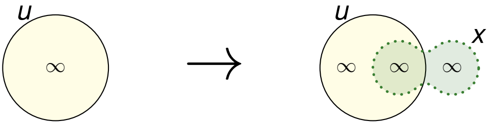
Set-theoretic mereology doesn't know any set theory.

The mereological model $\langle V, \subseteq\rangle$ doesn't know whether CH holds or fails, whether there are large cardinals, whether $V = L$ and so on for any of the huge variety of set-theoretic phenomenon.

Similarly, it doesn't detect differences in the arithmetic theory realized in models of set theory.

Role of the axiom of choice

We needed either the axiom of choice or ω-standardness for the saturation result.



Specifically, if a model of set theory has an *amorphous* set (and is ω-standard), then it will fail the saturation criterion.

Categoricity with amorphous sets?

Question

How many non-isomorphic countable models of set-theoretic mereology are there, if one allows amorphous sets?

This is a topic of current research.

Generalizing to classes

Most of the analysis generalizes directly to the usual second-order set theories, such as Gödel-Bernays set theory GBC or Kelley-Morse set theory KM.

Generalizing to classes

Most of the analysis generalizes directly to the usual second-order set theories, such as Gödel-Bernays set theory GBC or Kelley-Morse set theory KM.

Every model of second-order set theory gives rise to the class-theoretic mereological reduct $\langle M, \subseteq^M \rangle$. Keep all the classes, considered under the inclusion relation.

Generalizing to classes

Most of the analysis generalizes directly to the usual second-order set theories, such as Gödel-Bernays set theory GBC or Kelley-Morse set theory KM.

Every model of second-order set theory gives rise to the class-theoretic mereological reduct $\langle M, \subseteq^M \rangle$. Keep all the classes, considered under the inclusion relation.

This is an atomic Boolean algebra with infinitely many atoms.

Class-theoretic mereology

Our analysis extends to class theory to prove:

Theorem

If $\langle M, \in^M \rangle$ is a model of Gödel-Bernays class theory (considerably less suffices), then the corresponding inclusion relation $\langle M, \subseteq^M \rangle$ is an $\aleph_0$-saturated model of class-theoretic mereology, an $\aleph_0$-saturated infinite atomic Boolean algebra.

Class-theoretic mereology

Our analysis extends to class theory to prove:

Theorem

If $\langle M, \in^M \rangle$ is a model of Gödel-Bernays class theory (considerably less suffices), then the corresponding inclusion relation $\langle M, \subseteq^M \rangle$ is an $\aleph_0$-saturated model of class-theoretic mereology, an $\aleph_0$-saturated infinite atomic Boolean algebra.

We analyze the expressive power of types, and employ the Tarski quantifier-elimination argument.

Only one model

Corollary

All countable models of Gödel-Bernays class theory have the same inclusion relation, up to isomorphism. Specifically, if $\langle M, \in^M \rangle$ *and* $\langle N, \in^N \rangle$ *are each countable models of* GBC, *then* $\langle M, \subseteq^M \rangle$ *is isomorphic to* $\langle N, \subseteq^N \rangle$.

Proof.

Since $\langle M, \subseteq^M \rangle$ and $\langle N, \subseteq^N \rangle$ are each $\aleph_0$-saturated models of the same complete theory, they are isomorphic by the back-and-forth construction. □

Class-theoretic mereology is decidable

The decidability result also extends to class-theoretic mereology.

Class-theoretic mereology is decidable

The decidability result also extends to class-theoretic mereology.

The theory of Boolean algebras is decidable.

Class-theoretic mereology is decidable

The decidability result also extends to class-theoretic mereology.

The theory of Boolean algebras is decidable.

Thus, moving from sets to classes will not enable mereology to become a foundation of mathematics.

Summary

- Mereology is the study of the parthood relation
- Comparable in abstraction and generality to set theory
- Yet, mereology has not succeeded as a foundation of mathematics. Why is this?

Summary

- Mereology is the study of the parthood relation
- Comparable in abstraction and generality to set theory
- Yet, mereology has not succeeded as a foundation of mathematics. Why is this?
- I study the question via set-theoretic mereology $\subseteq$
- Set-theoretic mereology turns out to be: atomic unbounded relatively complemented distributive lattice

Summary

- Mereology is the study of the parthood relation
- Comparable in abstraction and generality to set theory
- Yet, mereology has not succeeded as a foundation of mathematics. Why is this?
- I study the question via set-theoretic mereology $\subseteq$
- Set-theoretic mereology turns out to be: atomic unbounded relatively complemented distributive lattice
- This is a decidable theory
- A decidable theory cannot provide a foundation of mathematics

Summary

- Mereology is the study of the parthood relation
- Comparable in abstraction and generality to set theory
- Yet, mereology has not succeeded as a foundation of mathematics. Why is this?
- I study the question via set-theoretic mereology $\subseteq$
- Set-theoretic mereology turns out to be: atomic unbounded relatively complemented distributive lattice
- This is a decidable theory
- A decidable theory cannot provide a foundation of mathematics
- Furthermore, all countable models of set theory have the same (isomorphic) mereological reduct
- Similar analysis and conclusions for class-theoretic mereology.

References

[Bau+85] A. Baudisch, D. Seese, P. Tuschik, and M. Weese. "Decidability and quantifier-elimination". In: *Model-theoretic Logics*. Perspect. Math. Logic. Springer, 1985, pp. 235–270.

[CK90] C. C. Chang and H. J. Keisler. *Model Theory*. Third. Vol. 73. Studies in Logic and the Foundations of Mathematics. North-Holland, 1990, pp. xvi+650. ISBN: 0-444-88054-2.

[Erš64] Ju. L. Eršov. "Decidability of the elementary theory of relatively complemented lattices and of the theory of filters". *Algebra i Logika Sem.* 3.3 (1964), pp. 17–38. ISSN: 0373-9252.

[Hil26] David Hilbert. "Über das Unendliche". *Mathematische Annalen* 95.1 (1926), pp. 161–190. ISSN: 0025-5831. DOI: 10.1007/BF01206605.

[HK16] Joel David Hamkins and Makoto Kikuchi. "Set-theoretic mereology". *Logic and Logical Philosophy, Special issue "Mereology and beyond, part II"* 25.3 (2016). Ed. by A. C. Varzi and R. Gruszczyński, pp. 285–308. ISSN: 1425-3305. DOI: 10.12775/LLP.2016.007. arXiv:1601.06593[math.LO]. http://jdh.hamkins.org/set-theoretic-mereology.

[HK17] Joel David Hamkins and Makoto Kikuchi. "The inclusion relations of the countable models of set theory are all isomorphic". *Mathematics arXiv* (2017). Manuscript under review. arXiv:1704.04480[math.LO]. http://jdh.hamkins.org/inclusion-relations-are-all-isomorphic.

[Hod93] Wilfrid Hodges. *Model Theory*. Vol. 42. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993, pp. xiv+772. ISBN: 0-521-30442-3. DOI: 10.1017/CBO9780511551574.

[Lew91] David Lewis. *Parts of Classes*. Blackwell, 1991. ISBN: 063117656X 063117656X pbk.

[Mon76] J. Donald Monk. *Mathematical Logic*. Graduate Texts in Mathematics, No. 37. Springer-Verlag, 1976, pp. x+531.

[Mos06] Yiannis Moschovakis. *Notes on Set Theory*. Undergraduate Texts in Mathematics. Springer, 2006. ISBN: 978-0387-28722-5.

[Poi00] Bruno Poizat. *A Course in Model Theory*. Universitext. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author. Springer-Verlag, 2000, pp. xxxii+443. ISBN: 0-387-98655-3. DOI: 10.1007/978-1-4419-8622-1.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
University of Notre Dame
Peking University