How the continuum hypothesis could have been fundamental

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

Irvine 15 March 2024

Georg Cantor proved that the real numbers form an *uncountable* infinity.

$$|\mathbb{N}| \quad < \quad |\mathbb{R}|$$

Georg Cantor proved that the real numbers form an *uncountable* infinity.

$$|\mathbb{N}| \quad < \quad |\mathbb{R}|$$

Is there any infinity between them?

Georg Cantor proved that the real numbers form an *uncountable* infinity.

$$|\mathbb{N}| \quad < \quad |\mathbb{R}|$$

Is there any infinity between them?

Continuum hypothesis (Cantor)

There is no infinity between $\mathbb{N}$ and $\mathbb{R}$.

In other words, CH asserts that the continuum is the first uncountable infinity.

$$|\mathbb{R}| = \mathfrak{c} = \beth_1 = 2^{\aleph_0} = |P(\mathbb{N})| \quad = \qquad \aleph_1.$$

A central open question

The continuum hypothesis was formulated by Georg Cantor in the late 19th century, and he spent his life in frustration trying to prove or refute it.

A central open question

The continuum hypothesis was formulated by Georg Cantor in the late 19th century, and he spent his life in frustration trying to prove or refute it.

The continuum problem was the first problem on Hilbert's famous list of open problems announced at the dawn of the 20th century, which went on to guide mathematical research since that time.

A central open question

The continuum hypothesis was formulated by Georg Cantor in the late 19th century, and he spent his life in frustration trying to prove or refute it.

The continuum problem was the first problem on Hilbert's famous list of open problems announced at the dawn of the 20th century, which went on to guide mathematical research since that time.

Cantor proved that it holds for closed sets, and his strategy of working up to more complicated sets is partially fulfilled by large cardinals.

Gödel

The CH question remained open for decades after Cantor.

Gödel: you cannot prove that CH is false

The CH question remained open for decades after Cantor.

In 1938, Kurt Gödel proved that you cannot refute the continuum hypothesis.

Gödel: you cannot prove that CH is false

The CH question remained open for decades after Cantor.

In 1938, Kurt Gödel proved that you cannot refute the continuum hypothesis.

He did so by proving that CH holds in the constructible universe L, a set-theoretic universe he described in which all the Zermelo-Fraenkel ZFC axioms are true, as well as CH.

If ZF is consistent, therefore, so is ZF plus the axiom of choice and the continuum hypothesis.

Cohen: you cannot prove CH is true

In 1963, Paul Cohen proved that you also cannot prove the continuum hypothesis.

Cohen: you cannot prove CH is true

In 1963, Paul Cohen proved that you also cannot prove the continuum hypothesis.

He did so by inventing the method of *forcing*, which allows one with any model of set theory M to construct a larger model of set theory $M[G]$ in which all the ZFC axioms remain true, yet CH fails.

Cohen: you cannot prove CH is true

In 1963, Paul Cohen proved that you also cannot prove the continuum hypothesis.

He did so by inventing the method of *forcing*, which allows one with any model of set theory M to construct a larger model of set theory $M[G]$ in which all the ZFC axioms remain true, yet CH fails.

The forcing extension $M[G]$ is something like a set-theoretic analogue of a field extension—we adjoin a new ideal object G, the generic filter, and every object in $M[G]$ is definable in a very concrete manner from an object in the ground model M and the new ideal object G.

CH is independent of ZFC

The continuum hypothesis is thus neither provable nor refutable in ZFC. It is *independent* of ZFC.

In fact, both CH and $\neg$CH are forceable over any model of ZFC.

CH is independent of ZFC

The continuum hypothesis is thus neither provable nor refutable in ZFC. It is *independent* of ZFC.

In fact, both CH and ¬CH are forceable over any model of ZFC.

Forcing has been used in thousands of mathematical arguments, revealing a pervasive ubiquity of the independence phenomenon.

CH is independent of ZFC

The continuum hypothesis is thus neither provable nor refutable in ZFC. It is *independent* of ZFC.

In fact, both CH and $\neg$CH are forceable over any model of ZFC.

Forcing has been used in thousands of mathematical arguments, revealing a pervasive ubiquity of the independence phenomenon.

Almost every nontrivial assertion of infinite combinatorics is independent of ZFC.

The Continuum problem

But is it true?

The Continuum problem

But is it true?

The independence of CH may simply be showing us the weakness of ZFC as a fundamental theory.

We may need to strengthen the underlying theory to settle CH.

CH and forcing

Both CH and ¬CH can be easily forced.

By moving successively to larger and larger set-theoretic worlds, we can turn CH on and off like a lightswitch.

CH not settled by large cardinals

Gödel had hoped that CH would be settled by strong axioms of infinity.

CH not settled by large cardinals

Gödel had hoped that CH would be settled by strong axioms of infinity.

But these hopes were dashed.

Theorem (Lévy+Solovay)

All of the commonly considered large cardinal hypotheses are preserved by the forcing of CH and of $\neg$CH.

We cannot use large cardinals to settle CH.

CH holds in the canonical inner models

Gödel's proof that CH holds in the constructible universe L has been extended to other canonical inner models.

It holds in the canonical inner model of a measurable cardinal $L[\mu]$, the extender models $L[\vec{E}]$, the core model K, and so forth.

CH is refuted by forcing axioms

The continuum hypothesis is refuted by the various forcing axioms.

Refuted by Martin's axiom MA_{ω_1}, the proper forcing axiom PFA, and Martin's Maximum MM.

The latter axioms prove that the continuum is

$$\mathfrak{c} = \aleph_2.$$

$\neg$CH routinely assumed in some areas

Researchers working on cardinal characteristics of the continuum routinely focus on $\neg\mathrm{CH}$, as the theory is trivialized under CH.

Philosophical attempts to settle CH

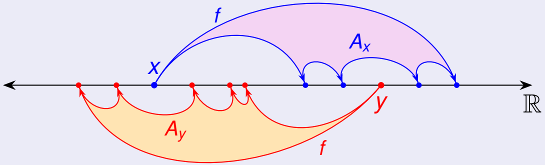
The truth or falsity of CH cannot be settled on the basis of proof from the ZFC axioms.

Set theorists have consequently offered various philosophical arguments aiming at a solution to the continuum problem, the problem of determining whether CH holds or its negation.

Freiling: throwing darts at the real line

The Axiom of Symmetry (Freiling JSL, 1986)

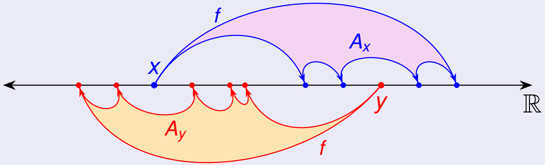
Asserts that for any function f mapping reals to countable sets of reals, there are x, y with $y \notin f(x)$ and $x \notin f(y)$.



Freiling: throwing darts at the real line

The Axiom of Symmetry (Freiling JSL, 1986)

Asserts that for any function f mapping reals to countable sets of reals, there are x, y with $y \notin f(x)$ and $x \notin f(y)$.



To justify the axiom, Freiling considers dart-throwing thought experiments.

Freiling: “a simple philosophical ‘proof’ of $\neg$CH”

Freiling then proves that the axiom of symmetry settles CH.

Freiling: “a simple philosophical ‘proof’ of $\neg\mathrm{CH}$”

Freiling then proves that the axiom of symmetry settles CH.

Theorem ([Fre86])

The axiom of symmetry is equivalent to $\neg\mathrm{CH}$.

Freiling: "a simple philosophical 'proof' of $\neg\mathrm{CH}$"

Freiling then proves that the axiom of symmetry settles CH.

Theorem ([Fre86])

The axiom of symmetry is equivalent to $\neg\mathrm{CH}$.

Higher-order versions, with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and indeed $f : \mathbb{R}^n \to \mathbb{R}$, are equivalent to the assertion $\mathfrak{c} > \aleph_n$.

Freiling: “a simple philosophical ‘proof’ of $\neg$CH”

Freiling then proves that the axiom of symmetry settles CH.

Theorem ([Fre86])

The axiom of symmetry is equivalent to $\neg$CH.

Higher-order versions, with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and indeed $f : \mathbb{R}^n \to \mathbb{R}$, are equivalent to the assertion $\mathfrak{c} > \aleph_n$.

The theorem was not generally accepted as a solution to CH, in light of non-measurable sets.

Similar attitudes toward Banach-Tarski in regard to AC.

Woodin

W. Hugh Woodin has advanced philosophical arguments on both sides of the CH debate.

Woodin

W. Hugh Woodin has advanced philosophical arguments on both sides of the CH debate.

- Woodin made a case for $\neg$CH based on considerations of Ω-logic and forcing absoluteness. [Koe23]

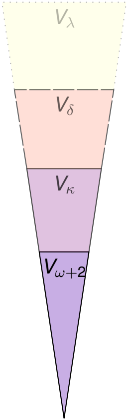
Woodin

W. Hugh Woodin has advanced philosophical arguments on both sides of the CH debate.

- Woodin made a case for $\neg\mathrm{CH}$ based on considerations of Ω-logic and forcing absoluteness. [Koe23]
- More recently, Woodin argues for CH based on features of his theory of Ultimate L, a canonical inner model accommodating even the largest large cardinals.

See [Rit15] for an account of Woodin's change of heart.

CH settled by second-order logic

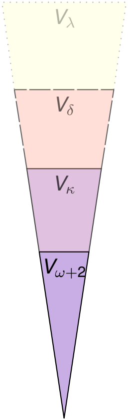


Kreisel [Kre67] argues (also Isaacson [Isa11]) that CH is settled in second-order logic.

Zermelo proved that the models of ZFC_2 are exactly V_κ for inaccessible cardinals κ. In particular, they agree a long way on the rank-initial segments V_α of the universe.

In particular, all the Zermelo universes agree on CH, which is revealed in $V_{\omega+2}$.

CH settled by second-order logic



Kreisel [Kre67] argues (also Isaacson [Isa11]) that CH is settled in second-order logic.

Zermelo proved that the models of ZFC_2 are exactly V_κ for inaccessible cardinals κ. In particular, they agree a long way on the rank-initial segments V_α of the universe.

In particular, all the Zermelo universes agree on CH, which is revealed in $V_{\omega+2}$.

Others criticize this observation as begging the question, or circular.

The very meaning of second-order logic is grounded in set theory.

CH at center of pluralism debate

The continuum hypothesis has often been at the center of the ongoing vigorous debate on pluralism taking place in the philosophy of set theory.

Universe view. According to this view, also known as *set-theoretic monism*, there is a unique absolute background concept of set, instantiated in the cumulative universe of all sets, in which set-theoretic assertions have a definite truth value.

Main challenge for the universe view

The central discovery in set theory over the past half-century is the enormous range of set-theoretic possibility. The most powerful set-theoretical tools are most naturally understood as methods of constructing alternative set-theoretical universes, universes that seem fundamentally set-theoretic.

forcing, ultrapowers, canonical inner models, etc.

Much of set-theory research has been about constructing as many different models of set theory as possible. These models are often made to exhibit precise, exacting features or to exhibit specific relationships with other models.

Set-theoretic pluralism

A competing position accepts the alternative set concepts as fully real.

The Multiverse view. Also known as *set-theoretic pluralism*, this is the philosophical position holding that there are numerous distinct legitimate concepts of set, each giving rise to a corresponding set-theoretic universe.

The dream solution

Many set theorists yearn for a *dream solution*, by which we settle CH by finding a “missing” axiom, which settles CH.

The dream solution

Many set theorists yearn for a *dream solution*, by which we settle CH by finding a “missing” axiom, which settles CH.

I argue in [Ham15] that this is impossible.

> *Our situation with* CH *is not merely that* CH *is formally independent and we have no additional knowledge about whether it is true or not. Rather, we have an informed, deep understanding of how it could be that* CH *is true and how it could be that* CH *fails. We know how to build the* CH *and* $\neg$CH *worlds from one another. Set theorists today grew up in these worlds, comparing them and moving from one to another while controlling other subtle features about them. Consequently, if someone were to present a new set-theoretic principle* Φ *and prove that it implies* $\neg$CH*, say, then we could no longer look upon* Φ *as manifestly true for sets. To do so would negate our experience in the* CH *worlds, which we found to be perfectly set-theoretic. It would be like someone proposing a principle implying that only Brooklyn really exists, whereas we already know about Manhattan and the other boroughs. And similarly if* Φ *were to imply* CH*. We are simply too familiar with universes exhibiting both sides of* CH *for us ever to accept as a natural set-theoretic truth a principle that is false in some of them.*

Is CH an open question?

I have argued that it is therefore incorrect to describe CH as an open question [Ham12].

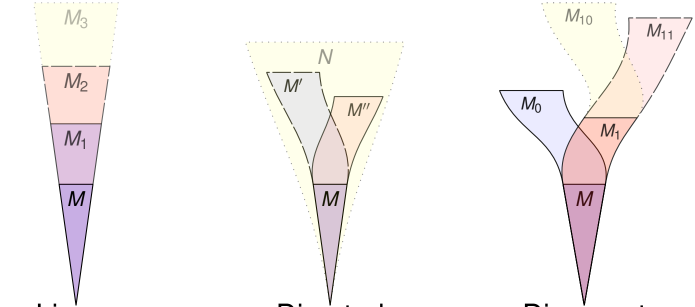
Is CH an open question?

I have argued that it is therefore incorrect to describe CH as an open question [Ham12].

Rather, the answer to CH consists of the deep body of knowledge that we have concerning how it behaves in the set-theoretic multiverse, how we can force it or its negation while preserving diverse other set-theoretic features.

Varieties of set-theoretic potentialism

A rich philosophical discussion of set-theoretic potentialism is emerging.



Linear inevitability S4.3 | Directed convergence S4.2 | Divergent potentialism S4

How it might have been different

We come now to the heart of my talk.

I should like to describe how our attitude toward the continuum hypothesis could easily have been very different than it is.

How it might have been different

We come now to the heart of my talk.

I should like to describe how our attitude toward the continuum hypothesis could easily have been very different than it is.

Namely, if our mathematical history had been just a little different, I claim, if certain mathematical discoveries had been made in a slightly different order, then we would naturally view the continuum hypothesis as a fundamental axiom of set theory, one furthermore necessary for mathematics and indeed indispensable for calculus.

The thought experiment

Let us imagine that in the early days of calculus, Newton and Leibinz had provided somewhat fuller accounts of their ideas about infinitesimals.

The thought experiment

Let us imagine that in the early days of calculus, Newton and Leibinz had provided somewhat fuller accounts of their ideas about infinitesimals.

In the actual world a satisfactory account was lacking. The infinitesimal foundations were mocked by Berkeley [Ber34]:

> *And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?*

It was simply not clear enough what kind of thing the infinitesimals were—were they part of the ordinary number system or did they somehow transcend it?

Two realms of numbers

Let us imagine that Newton and Leibniz conceive of the infinitesimals as living in a larger field of numbers, distinct from but extending the ordinary real numbers.

Two realms of numbers

Let us imagine that Newton and Leibniz conceive of the infinitesimals as living in a larger field of numbers, distinct from but extending the ordinary real numbers.

Two "realms" of numbers:

- The real numbers $\mathbb{R}$
- A larger realm of numbers $\mathbb{R}^*$, let us call them *hyperreal*

This proposal immediately addresses the withering Berkeley criticism.

Clarifying nature of infinitesimals

The proposal of two number realms immediately and greatly clarifies much of the early discussion of infinitesimals.

- Immediately releases tension of paradoxical claim that infinitesimals are positive, yet smaller than every positive number

Clarifying nature of infinitesimals

The proposal of two number realms immediately and greatly clarifies much of the early discussion of infinitesimals.

- Immediately releases tension of paradoxical claim that infinitesimals are positive, yet smaller than every positive number
- Enables us to clarify more precisely how the infinitesimals relate to the real numbers.

Clarifying nature of infinitesimals

The proposal of two number realms immediately and greatly clarifies much of the early discussion of infinitesimals.

- Immediately releases tension of paradoxical claim that infinitesimals are positive, yet smaller than every positive number
- Enables us to clarify more precisely how the infinitesimals relate to the real numbers.
- Enables a frank discussion of the nature of the real numbers and infinitesimals

Two number systems, same laws and truths

To justify his calculations with infinitesimals, I imagine Leibniz writing:

> *"The two number realms fulfill all the same fundamental mathematical laws."*

Two number systems, same laws and truths

To justify his calculations with infinitesimals, I imagine Leibniz writing:

> *"The two number realms fulfill all the same fundamental mathematical laws."*

The hyperreal numbers thus fulfill the associativity and distributivity laws, and indeed any law that is true for the real numbers.

This same-laws view justifies the common calculations with infinitesimals that one finds in calculus.

Incipient tranfer principle

Imaginary Leibniz

"The two number realms fulfill all the same fundamental mathematical laws."

Incipient tranfer principle

Imaginary Leibniz

"The two number realms fulfill all the same fundamental mathematical laws."

From contemporary perspective, this is an incipient form of:

Transfer principle

All structure on the real numbers $\mathbb{R}$ extends to the hyperreal numbers with the same truth.

$$\langle \mathbb{R}, +, \cdot, 0, 1, <, \mathbb{Z}, f, \ldots \rangle \prec \langle \mathbb{R}^*, +, \cdot, 0, 1, <, \mathbb{Z}^*, f^*, \ldots \rangle$$

Incipient tranfer principle

Imaginary Leibniz

"The two number realms fulfill all the same fundamental mathematical laws."

From contemporary perspective, this is an incipient form of:

Transfer principle

All structure on the real numbers $\mathbb{R}$ extends to the hyperreal numbers with the same truth.

$$\langle \mathbb{R}, +, \cdot, 0, 1, <, \mathbb{Z}, f, \ldots \rangle \prec \langle \mathbb{R}^*, +, \cdot, 0, 1, <, \mathbb{Z}^*, f^*, \ldots \rangle$$

Thus, $\mathbb{R}^*$ is an ordered field, a real-closed field; every positive number has square root; every odd-degree polynomial has root.

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

- Explains nature and existence of infinitesimals

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

- Explains nature and existence of infinitesimals
- Gap between 0 and positive reals is filled by infinitesimals

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

“Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals.”

- Explains nature and existence of infinitesimals
- Gap between 0 and positive reals is filled by infinitesimals
- Hyperreal numbers at infinitesimal distance to $\sqrt{2}$, and to π

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

- Explains nature and existence of infinitesimals
- Gap between 0 and positive reals is filled by infinitesimals
- Hyperreal numbers at infinitesimal distance to $\sqrt{2}$, and to π
- Every real number has an infinitesimal neighborhood

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

- Explains nature and existence of infinitesimals
- Gap between 0 and positive reals is filled by infinitesimals
- Hyperreal numbers at infinitesimal distance to $\sqrt{2}$, and to π
- Every real number has an infinitesimal neighborhood
- Reciprocals of infinitesimals are infinite hyperreal numbers.

Infinitesimal existence

Let us also imagine Newton clarifying the existence of infinitesimals.

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

- Explains nature and existence of infinitesimals
- Gap between 0 and positive reals is filled by infinitesimals
- Hyperreal numbers at infinitesimal distance to $\sqrt{2}$, and to π
- Every real number has an infinitesimal neighborhood
- Reciprocals of infinitesimals are infinite hyperreal numbers.
- Hyperreal field is not Archimedian.

Filling gaps

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

Filling gaps

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ *is filled by infinitesimals."*

Contemporary perspective: this is incipient form of *saturation*.

Filling gaps

Imaginary Newton

"Every conceivable gap in $\mathbb{R}$ is filled by infinitesimals."

Contemporary perspective: this is incipient form of *saturation*.

Hyperreal field $\mathbb{R}^*$ is countably saturated

Every countable gap in $\mathbb{R}^*$

$$x_0 < x_1 < x_2 < \cdots \qquad\qquad \cdots < y_2 < y_1 < y_0$$

is filled by a hyperreal number z

$$x_0 < x_1 < x_2 < \cdots \quad < z < \quad \cdots < y_2 < y_1 < y_0$$

And same for one-sided gaps.

A modest proposal

The thought experiment is merely that Newton and Leibniz have expressed the primitive idea of two distinct number realms, with vaguely expressed ideas that we may now view as incipient forms of the transfer and saturation principles.

A modest proposal

The thought experiment is merely that Newton and Leibniz have expressed the primitive idea of two distinct number realms, with vaguely expressed ideas that we may now view as incipient forms of the transfer and saturation principles.

The proposal is not that they somehow have a full-blown well-formulated theory of the hyperreal numbers.

Rather, further development and rigor will naturally come in time, just as it did in our actual mathematical history.

Fundamentally coherent account of infinitesimals

We know now that transfer and saturation are fundamentally coherent and correct accounts of the hyperreal numbers.

Fundamentally coherent account of infinitesimals

We know now that transfer and saturation are fundamentally coherent and correct accounts of the hyperreal numbers.

Furthermore, such ideas are sufficient for a highly successful, insightful development of all the fundamental theory of calculus.

Fundamentally coherent account of infinitesimals

We know now that transfer and saturation are fundamentally coherent and correct accounts of the hyperreal numbers.

Furthermore, such ideas are sufficient for a highly successful, insightful development of all the fundamental theory of calculus.

See [Kei00] for an example of how one can develop the whole theory on the basis of very primitive notions.

Robust development of infinitesimal calculus

In our actual history, even an incoherent account of infinitesimals was highly successful and led to many insightful discoveries, including all the fundamental theorems of calculus.

Robust development of infinitesimal calculus

In our actual history, even an incoherent account of infinitesimals was highly successful and led to many insightful discoveries, including all the fundamental theorems of calculus.

Question

Does one need rigorous foundations for insightful mathematical discoveries of enduring importance?

Robust development of infinitesimal calculus

In our actual history, even an incoherent account of infinitesimals was highly successful and led to many insightful discoveries, including all the fundamental theorems of calculus.

Question

Does one need rigorous foundations for insightful mathematical discoveries of enduring importance?

Apparently not.

Robust development of infinitesimal calculus

In our actual history, even an incoherent account of infinitesimals was highly successful and led to many insightful discoveries, including all the fundamental theorems of calculus.

Question

Does one need rigorous foundations for insightful mathematical discoveries of enduring importance?

Apparently not.

Meanwhile, in the thought experiment, all the actual insights would be made and more, with increasing rigor and sophistication—an enduring calculus based on infinitesimals, proceeding roughly along the lines of nonstandard analysis.

Hyperreal numbers as a familiar number system

At bottom, the proposal is that the hyperreal numbers would have become one of the standard number systems that mathematicians discovered and became familiar with.

$$\mathbb{N} \qquad \mathbb{Z} \qquad \mathbb{Q} \qquad \mathbb{R} \qquad \mathbb{C}$$

Hyperreal numbers as a familiar number system

At bottom, the proposal is that the hyperreal numbers would have become one of the standard number systems that mathematicians discovered and became familiar with.

$$\mathbb{N} \qquad \mathbb{Z} \qquad \mathbb{Q} \qquad \mathbb{R} \qquad \mathbb{C} \qquad \mathbb{R}^*$$

We would find the hyperreal numbers alongside the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers, serving as one of the familiar standard number systems.

Categoricity

At the end of the nineteenth and early twentieth century, mathematicians began to provide *categorical* accounts of all our most fundamental number systems.

Categoricity

At the end of the nineteenth and early twentieth century, mathematicians began to provide *categorical* accounts of all our most fundamental number systems.

A *categorical* account of a mathematical structure identifies axioms true in that structure, such that those axioms furthermore determine that structure up to isomorphism.

Categorical accounts generally use second-order logic (impossible in first-order logic).

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle\mathbb{N}, 0, S\rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle \mathbb{N}, 0, S \rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]
- Using this, mathematicians provide categorical accounts of the integer ring $\mathbb{Z}$ and the rational field $\mathbb{Q}$.

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle \mathbb{N}, 0, S \rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]
- Using this, mathematicians provide categorical accounts of the integer ring $\mathbb{Z}$ and the rational field $\mathbb{Q}$.
- Cantor proves that the rational order $\langle \mathbb{Q}, < \rangle$ is characterized as the unique countable endless dense linear order. [Can95; Can97; Can52]

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle \mathbb{N}, 0, S \rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]
- Using this, mathematicians provide categorical accounts of the integer ring $\mathbb{Z}$ and the rational field $\mathbb{Q}$.
- Cantor proves that the rational order $\langle \mathbb{Q}, < \rangle$ is characterized as the unique countable endless dense linear order. [Can95; Can97; Can52]
- Huntington provides the categorical account of the real field $\mathbb{R}$ as the unique complete ordered field. [Hun03]

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle \mathbb{N}, 0, S\rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]
- Using this, mathematicians provide categorical accounts of the integer ring $\mathbb{Z}$ and the rational field $\mathbb{Q}$.
- Cantor proves that the rational order $\langle \mathbb{Q}, <\rangle$ is characterized as the unique countable endless dense linear order. [Can95; Can97; Can52]
- Huntington provides the categorical account of the real field $\mathbb{R}$ as the unique complete ordered field. [Hun03]
- The complex numbers are characterized as the algebraic closure of $\mathbb{R}$.

Categorical accounts of the central structures of mathematics

- Dedekind proves that the natural number structure $\langle\mathbb{N}, 0, S\rangle$ is uniquely specified up to isomorphism by his theory of the successor operation. [Ded88]
- Using this, mathematicians provide categorical accounts of the integer ring $\mathbb{Z}$ and the rational field $\mathbb{Q}$.
- Cantor proves that the rational order $\langle\mathbb{Q}, <\rangle$ is characterized as the unique countable endless dense linear order. [Can95; Can97; Can52]
- Huntington provides the categorical account of the real field $\mathbb{R}$ as the unique complete ordered field. [Hun03]
- The complex numbers are characterized as the algebraic closure of $\mathbb{R}$.

Each of the central structures in mathematics enjoys a clear categorical characterization.

Importance of categoricity

Categorical accounts for our central mathematical structures enables a necessary coherence of the mathematical enterprise.

Categoricity enables us to refer to the various fundamental mathematical structures by their defining characteristics.

Importance of categoricity

Categorical accounts for our central mathematical structures enables a necessary coherence of the mathematical enterprise.

Categoricity enables us to refer to the various fundamental mathematical structures by their defining characteristics.

Categoricity also implements, in a direct practical manner, the philosophy of structuralism, by which we treat all our mathematical structures and features as invariant under isomorphism.

Importance of categoricity

Categorical accounts for our central mathematical structures enables a necessary coherence of the mathematical enterprise.

Categoricity enables us to refer to the various fundamental mathematical structures by their defining characteristics.

Categoricity also implements, in a direct practical manner, the philosophy of structuralism, by which we treat all our mathematical structures and features as invariant under isomorphism.

Two independent strands of structuralism

Importance of categoricity

Categorical accounts for our central mathematical structures enables a necessary coherence of the mathematical enterprise.

Categoricity enables us to refer to the various fundamental mathematical structures by their defining characteristics.

Categoricity also implements, in a direct practical manner, the philosophy of structuralism, by which we treat all our mathematical structures and features as invariant under isomorphism.

Two independent strands of structuralism

- A philosophical strand, growing out of Benacerraf.

Importance of categoricity

Categorical accounts for our central mathematical structures enables a necessary coherence of the mathematical enterprise.

Categoricity enables us to refer to the various fundamental mathematical structures by their defining characteristics.

Categoricity also implements, in a direct practical manner, the philosophy of structuralism, by which we treat all our mathematical structures and features as invariant under isomorphism.

Two independent strands of structuralism

- A philosophical strand, growing out of Benacerraf.
- A mathematical strand, tracing to Dedekind categoricity.

Isaacson on categoricity

Particular structures are found by mathematical experience, and then characterized as unique.

> *"If the mathematical community at some stage in the development of mathematics has succeeded in becoming (informally) clear about a particular mathematical structure, this clarity can be made mathematically exact... usually by means of a full second-order language. Why must there be such a characterisation? Answer: if the clarity is genuine, there must be a way to articulate it precisely. If there is no such way, the seeming clarity must be illusory. Such a claim is of the character as the Church-Turing thesis,... for every particular structure developed in the practice of mathematics, there is [a] categorical characterization of it."(p. 31, Reality of Mathematics...)*

In our thought experiment, mathematicians would have become informally clear about the hyperreal field structure. The situation would call out for a categorical characterization.

Hyperreal categoricity

Is it possible? Can we have a categorical account the hyperreals $\mathbb{R}^*$?

Hyperreal categoricity

Is it possible? Can we have a categorical account the hyperreals $\mathbb{R}^*$?

In the imaginary history, the hyperreal numbers $\mathbb{R}^*$ have become a core mathematical conception, present from the beginning at the foundations of calculus, and mathematicians would insist upon a categorical account.

The key event

Yes, we can give a categorical account of the hyperreal numbers.

The key event

Yes, we can give a categorical account of the hyperreal numbers.

The key event

Continuing our thought experiment, let us imagine that a Zermelo-like figure formulates a theory—the theory ZFC $+\,\mathrm{CH}$ suffices—able to prove a categorical characterization of the hyperreal number field $\mathbb{R}^*$.

The key event

Yes, we can give a categorical account of the hyperreal numbers.

The key event

Continuing our thought experiment, let us imagine that a Zermelo-like figure formulates a theory—the theory ZFC $+ \mathrm{CH}$ suffices—able to prove a categorical characterization of the hyperreal number field $\mathbb{R}^*$.

This would be similar to how actual Zermelo provided the ZC axioms as explanation for his proof of the well-order theorem.

It was a formative time, when our foundational theories were first articulated.

CH suffices for hyperreal categoricity

We can prove hyperreal categoricity from the continuum hypothesis.

Theorem

Assume ZFC + CH. *Then there is up to isomorphism a unique countably saturated ordered field extension of* $\mathbb{R}$ *of size continuum with the transfer principle.*

CH suffices for hyperreal categoricity

We can prove hyperreal categoricity from the continuum hypothesis.

Theorem

Assume ZFC + CH. *Then there is up to isomorphism a unique countably saturated ordered field extension of* $\mathbb{R}$ *of size continuum with the transfer principle.*

This can be proved in a back-and-forth argument, much like Cantor's DLO argument, but with transfinite length ω_1.

Sharper forms of hyperreal categoricity

In fact, we don't need the full transfer principle, and only need saturation for the order.

Sharper forms of hyperreal categoricity

In fact, we don't need the full transfer principle, and only need saturation for the order.

Theorem (Erdős, Gillman, and Henriksen [EGH55])

Under CH there is up to isomorphism only one real-closed field of size continuum whose order is countably saturated.

Sharper forms of hyperreal categoricity

In fact, we don't need the full transfer principle, and only need saturation for the order.

Theorem (Erdős, Gillman, and Henriksen [EGH55])

Under CH there is up to isomorphism only one real-closed field of size continuum whose order is countably saturated.

These hypotheses are close to the original principles we had attributed to our imaginary Newton and Leibniz.

In this way, from CH we can prove there is a unique hyperreal structure up to isomorphism.

CH is required

Meanwhile, there is no categoricity result without CH.

CH is required

Meanwhile, there is no categoricity result without CH.

- Roitman [Roi82] shows it is relatively consistent with ZFC $+\neg\mathrm{CH}$ to have multiple non-isomorphic hyperreal fields arising as ultrapowers $\mathbb{R}^\omega/\mu$.

CH is required

Meanwhile, there is no categoricity result without CH.

- Roitman [Roi82] shows it is relatively consistent with ZFC $+ \neg\mathrm{CH}$ to have multiple non-isomorphic hyperreal fields arising as ultrapowers $\mathbb{R}^\omega/\mu$.
- Alan Dow [Dow84] shows that whenever CH fails, then indeed there are multiple non-isomorphic ultrapowers $\mathbb{R}^\omega/\mu$, non-isomorphic even merely in their order structure.

CH is required

Meanwhile, there is no categoricity result without CH.

- Roitman [Roi82] shows it is relatively consistent with ZFC $+ \neg\mathrm{CH}$ to have multiple non-isomorphic hyperreal fields arising as ultrapowers $\mathbb{R}^\omega/\mu$.
- Alan Dow [Dow84] shows that whenever CH fails, then indeed there are multiple non-isomorphic ultrapowers $\mathbb{R}^\omega/\mu$, non-isomorphic even merely in their order structure.
- Thus, CH iff there is a unique countably saturated real-closed field of size continuum.

CH is required

Meanwhile, there is no categoricity result without CH.

- Roitman [Roi82] shows it is relatively consistent with ZFC $+ \neg\mathrm{CH}$ to have multiple non-isomorphic hyperreal fields arising as ultrapowers $\mathbb{R}^\omega/\mu$.
- Alan Dow [Dow84] shows that whenever CH fails, then indeed there are multiple non-isomorphic ultrapowers $\mathbb{R}^\omega/\mu$, non-isomorphic even merely in their order structure.
- Thus, CH iff there is a unique countably saturated real-closed field of size continuum.

In short,

- With CH, we have categoricity for the hyperreals.

CH is required

Meanwhile, there is no categoricity result without CH.

- Roitman [Roi82] shows it is relatively consistent with ZFC $+ \neg\mathrm{CH}$ to have multiple non-isomorphic hyperreal fields arising as ultrapowers $\mathbb{R}^\omega/\mu$.
- Alan Dow [Dow84] shows that whenever CH fails, then indeed there are multiple non-isomorphic ultrapowers $\mathbb{R}^\omega/\mu$, non-isomorphic even merely in their order structure.
- Thus, CH iff there is a unique countably saturated real-closed field of size continuum.

In short,

- With CH, we have categoricity for the hyperreals.
- Without CH, we lack categoricity for the hyperreals.

“The hyperreal numbers” not meaningful in ZFC

ZFC does not prove a unique hyperreal structure.

“The hyperreal numbers” not meaningful in ZFC

ZFC does not prove a unique hyperreal structure.

I have argued that lack of categoricity for $\mathbb{R}^*$ explains hesitancy for nonstandard analysis amongst mathematicians. [Ham21]

Mathematicians are loathe to mount a fundamental theory with underspecified structures at core.

“The hyperreal numbers” not meaningful in ZFC

ZFC does not prove a unique hyperreal structure.

I have argued that lack of categoricity for $\mathbb{R}^*$ explains hesitancy for nonstandard analysis amongst mathematicians. [Ham21]

Mathematicians are loathe to mount a fundamental theory with underspecified structures at core.

If multiple structures, which one do we use? How can we even describe which one?

Lack of categoricity $\Longrightarrow$ lack of reference.

"The hyperreal numbers" not meaningful in ZFC

ZFC does not prove a unique hyperreal structure.

I have argued that lack of categoricity for $\mathbb{R}^*$ explains hesitancy for nonstandard analysis amongst mathematicians. [Ham21]

Mathematicians are loathe to mount a fundamental theory with underspecified structures at core.

If multiple structures, which one do we use? How can we even describe which one?

Lack of categoricity $\implies$ lack of reference.

Categoricity is required for a coherent structuralist practice.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.
- Pre-rigorous, but then with increasing rigor, sophistication.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.
- Pre-rigorous, but then with increasing rigor, sophistication.
- Categoricity required for coherent mathematical practice.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.
- Pre-rigorous, but then with increasing rigor, sophistication.
- Categoricity required for coherent mathematical practice.
- Zermelo-like figure introduces ZFC $+ \mathrm{CH}$ to prove categoricity.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.
- Pre-rigorous, but then with increasing rigor, sophistication.
- Categoricity required for coherent mathematical practice.
- Zermelo-like figure introduces ZFC $+ \text{CH}$ to prove categoricity.
- We know that CH is required for this.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

- Hyperreal field $\mathbb{R}^*$ becomes a core mathematical idea, present from the beginning.
- Pre-rigorous, but then with increasing rigor, sophistication.
- Categoricity required for coherent mathematical practice.
- Zermelo-like figure introduces ZFC $+ \mathrm{CH}$ to prove categoricity.
- We know that CH is required for this.

Thus, CH becomes necessary part of the foundational theory establishing the basic coherence of the hyperreal numbers.

CH becomes indispensable for the foundations of calculus.

Extrinsic justification of CH

The categoricity account of $\mathbb{R}^*$ provides enormous extrinsic support for CH.

Extrinsic justification of CH

The categoricity account of $\mathbb{R}^*$ provides enormous extrinsic support for CH.

Similar to the current justification of ZFC in light of its successful account of the real numbers $\mathbb{R}$.

CH would be seen as vital for the account of the hyperreals $\mathbb{R}^*$, a core mathematical structure in the thought-experiment world.

Intrinsic justification

After the extrinsic justification, CH will naturally find intrinsic justification.

Intrinsic justification

After the extrinsic justification, CH will naturally find intrinsic justification.

Similar to how axiom of choice is viewed at first as extrinsically justified, then often intrinsically.

Intrinsic justification

After the extrinsic justification, CH will naturally find intrinsic justification.

Similar to how axiom of choice is viewed at first as extrinsically justified, then often intrinsically.

For CH, it asserts agreement for the two known methods of achieving uncountability.

Intrinsic justification

After the extrinsic justification, CH will naturally find intrinsic justification.

Similar to how axiom of choice is viewed at first as extrinsically justified, then often intrinsically.

For CH, it asserts agreement for the two known methods of achieving uncountability. $\aleph_1 = \beth_1$.

This is a unifying, explanatory principle of the uncountable, therefore intrinsic justification of CH.

The aftermath

In the imaginary history. . .

- Gödel proves that ZFC+CH is true in the constructible universe L. Welcome confirmation of theory ZFC $+$ CH.

The aftermath

In the imaginary history...

- Gödel proves that ZFC+CH is true in the constructible universe L. Welcome confirmation of theory ZFC $+$ CH.
- Extrinsic support for $V = L$.

The aftermath

In the imaginary history...

- Gödel proves that ZFC+CH is true in the constructible universe L. Welcome confirmation of theory ZFC $+$ CH.
- Extrinsic support for $V = L$.
- Categoricity generalizes under GCH to higher cardinalities.

The aftermath

In the imaginary history. . .

- Gödel proves that ZFC+CH is true in the constructible universe L. Welcome confirmation of theory ZFC $+\ \mathrm{CH}$.
- Extrinsic support for $V = L$.
- Categoricity generalizes under GCH to higher cardinalities.
- View of higher-cardinal hyperreal fields converging to the surreal numbers.

The aftermath

In the imaginary history...

- Gödel proves that ZFC+CH is true in the constructible universe L. Welcome confirmation of theory ZFC $+ \mathrm{CH}$.
- Extrinsic support for $V = L$.
- Categoricity generalizes under GCH to higher cardinalities.
- View of higher-cardinal hyperreal fields converging to the surreal numbers.
- Surreal numbers categoricity result under global choice.

Forcing

The discovery via forcing that without CH there can be multiple non-isomorphic hyperreal fields would be seen as chaotic and bizarre.

Forcing

The discovery via forcing that without CH there can be multiple non-isomorphic hyperreal fields would be seen as chaotic and bizarre.

It would be perhaps like current attitudes about models of ZF with strange failures of the axiom of choice. For example, non-isomorphic algebraic closures of $\mathbb{Q}$. Often considered weird.

Forcing

The discovery via forcing that without CH there can be multiple non-isomorphic hyperreal fields would be seen as chaotic and bizarre.

It would be perhaps like current attitudes about models of ZF with strange failures of the axiom of choice. For example, non-isomorphic algebraic closures of $\mathbb{Q}$. Often considered weird.

Similarly odd to have multiple non-isomorphic hyperreal fields, reinforcing view that ZFC $+\ \mathrm{CH}$ is the right theory.

Different view of forcing

In the imaginary universe, forcing would be received differently than in our world.

Different view of forcing

In the imaginary universe, forcing would be received differently than in our world.

Forcing would be seen as less successful, since CH not preserved.

Different view of forcing

In the imaginary universe, forcing would be received differently than in our world.

Forcing would be seen as less successful, since CH not preserved.

In actual world, major attraction of forcing is that it preserves the fundamental theory ZFC. No longer true in imaginary world, since CH not preserved.

Forcing viewed like symmetric model construction—a means to produce weird counterexample models. Unnatural without CH.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

Early mathematicians could have been clearer about infinitesimals, positing distinct realms of numbers.

The hyperreal numbers $\mathbb{R}^*$ would have become a core mathematical structure.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

Early mathematicians could have been clearer about infinitesimals, positing distinct realms of numbers.

The hyperreal numbers $\mathbb{R}^*$ would have become a core mathematical structure.

All core structures require a categorical characterization.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

Early mathematicians could have been clearer about infinitesimals, positing distinct realms of numbers.

The hyperreal numbers $\mathbb{R}^*$ would have become a core mathematical structure.

All core structures require a categorical characterization.

But a cateogorical account of $\mathbb{R}^*$ is possible only with CH.

So CH would have been a part of our foundational theory. Extrinsically justified, but then also intrinsically.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

Early mathematicians could have been clearer about infinitesimals, positing distinct realms of numbers.

The hyperreal numbers $\mathbb{R}^*$ would have become a core mathematical structure.

All core structures require a categorical characterization.

But a cateogorical account of $\mathbb{R}^*$ is possible only with CH.

So CH would have been a part of our foundational theory.
Extrinsically justified, but then also intrinsically.

We would have viewed CH as necessary for mathematics, indispensible even for calculus.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

References I

[Ber34] George Berkeley. *A Discourse Addressed to an Infidel Mathematician*. The Strand, 1734. https://en.wikisource.org/wiki/The_Analyst:_a_Discourse_addressed_to_an_Infidel_Mathematician.

[Can52] Georg Cantor. *Contributions to the founding of the theory of transfinite numbers*. Translated, and provided with an introduction and notes, by Philip E. B. Jourdain. Dover Publications, Inc., New York, 1952, pp. ix+211.

[Can95] Georg Cantor. "Beiträge zur Begründung der transfiniten Mengenlehre". *Mathematische Annalen* 46 (1895). (German), pp. 481–512. DOI: 10.1007/BF02124929.

[Can97] Georg Cantor. "Beiträge zur Begründung der transfiniten Mengenlehre". *Mathematische Annalen* 49 (1897). (German), pp. 207–246. DOI: 10.1007/BF01444205.

References II

[Ded88] Richard Dedekind. "Was sind und was sollen die Zahlen? (What are numbers and what should they be?)" (1888). Available in Ewald, William B. 1996. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Vol. 2, 787–832. Oxford University Press, pp. 787–832.

[Dow84] Alan Dow. "On ultra powers of Boolean algebras". *Topology Proceedings* 9.2 (1984), pp. 269–291.

[EGH55] P. Erdős, L. Gillman, and M. Henriksen. "An isomorphism theorem for real-closed fields". *Ann. of Math. (2)* 61 (1955), pp. 542–554. ISSN: 0003-486X. DOI: 10.2307/1969812. https://doi.org/10.2307/1969812.

[Fre86] Chris Freiling. "Axioms of symmetry: throwing darts at the real number line". *J. Symbolic Logic* 51.1 (1986), pp. 190–200. ISSN: 0022-4812. DOI: 10.2307/2273955.

References III

[Ham12] Joel David Hamkins. “The set-theoretic multiverse”. *Review of Symbolic Logic* 5 (2012), pp. 416–449. DOI: 10.1017/S1755020311000359. arXiv:1108.4223[math.LO]. http://jdh.hamkins.org/themultiverse.

[Ham15] Joel David Hamkins. “Is the dream solution of the continuum hypothesis attainable?” *Notre Dame Journal of Formal Logic* 56.1 (2015), pp. 135–145. ISSN: 0029-4527. DOI: 10.1215/00294527-2835047. arXiv:1203.4026[math.LO]. http://jdh.hamkins.org/dream-solution-of-ch.

[Ham21] Joel David Hamkins. *Lectures on the Philosophy of Mathematics*. MIT Press, 2021. ISBN: 9780262542234. https://mitpress.mit.edu/books/lectures-philosophy-mathematics.

[Hun03] Edward V. Huntington. “Complete Sets of Postulates for the Theory of Real Quantities”. *Transactions of the American Mathematical Society* 4.3 (1903), pp. 358–370. ISSN: 00029947. http://www.jstor.org/stable/1986269.

References IV

[Isa11] Daniel Isaacson. "The reality of mathematics and the case of set theory". In: *Truth, Reference, and Realism*. Ed. by Zsolt Novak and Andras Simonyi. Central European University Press, 2011, pp. 1–76.

[Kei00] H. Jerome Keisler. *Elementary Calculus: An Infinitesimal Approach*. Earlier editions 1976, 1986 by Prindle, Weber, and Schmidt; free electronic edition available. 2000. https://www.math.wisc.edu/~keisler/calc.html.

[Koe23] Peter Koellner. "The Continuum Hypothesis". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta and Uri Nodelman. Winter 2023. Metaphysics Research Lab, Stanford University, 2023.

[Kre67] G. Kreisel. "Informal Rigour and Completeness Proofs [with Discussion]". In: *Problems in the Philosophy of Mathematics*. Ed. by Imre Lakatos. North-Holland, 1967, pp. 138–186.

References V

[Rit15] Colin J. Rittberg. “How Woodin changed his mind: new thoughts on the Continuum Hypothesis”. *Archive for History of Exact Sciences* 69.2 (2015), pp. 125–151. ISSN: 00039519, 14320657. http://www.jstor.org/stable/24569622 (version 20 January 2024).

[Roi82] J. Roitman. “Non-isomorphic hyper-real fields from non-isomorphic ultrapowers”. *Mathematische Zeitschrift* 181 (1982), pp. 93–96. DOI: 10.1007/BF01214984.