How we might have taken the continuum hypothesis as a fundamental axiom, necessary for mathematics

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William Nelson Reinhardt

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In memory (1939–1998)
(College of Wooster, 1961 yearbook)

The continuum hypothesis

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Continuum hypothesis (Cantor)

There is no infinity between $\mathbb{N}$ and $\mathbb{R}$.

In other words, CH asserts that the continuum is the first uncountable infinity.

$$|\mathbb{R}| = \mathfrak{c} = \beth_1 = 2^{\aleph_0} = |P(\mathbb{N})| \quad = \qquad \aleph_1.$$

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Cantor proved that it holds for closed sets, and his strategy of working up to more complicated sets is partially fulfilled by consequences of large cardinals in descriptive set theory.

Gödel

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In 1938, Kurt Gödel proved that you cannot refute the continuum hypothesis.

He did so by proving that CH holds in the constructible universe L, a set-theoretic universe he described in which all the Zermelo-Fraenkel ZFC axioms are true, as well as CH.

If ZF is consistent, therefore, so is ZF plus the axiom of choice and the continuum hypothesis.

Cohen: you cannot prove CH is true

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In 1963, Paul Cohen proved that you also cannot prove the continuum hypothesis.

He did so by inventing the method of *forcing*, which allows one with any model of set theory M to construct a larger model of set theory $M[G]$ in which all the ZFC axioms remain true, yet CH fails.

CH is independent of ZFC

The continuum hypothesis is thus neither provable nor refutable in ZFC. It is *independent* of ZFC.

In fact, both CH and $\neg$CH are forceable over any model of ZFC. We can turn it on and off in subsequent extensions like a lightswitch.

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Forcing has been used in thousands of mathematical arguments, revealing a pervasive ubiquity of the independence phenomenon.

Almost every nontrivial assertion of infinite combinatorics is independent of ZFC.

The Continuum problem

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But is it true?

The independence of CH may simply be showing us the weakness of ZFC as a fundamental theory.

We may need to strengthen the underlying theory to settle CH.

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But these hopes were dashed.

Theorem (Lévy+Solovay)

All of the commonly considered large cardinal hypotheses are preserved by the forcing of CH and of $\neg$CH.

We cannot use large cardinals to settle CH.

CH in set theory

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- CH refuted by forcing axioms MA_{ω_1}, PFA, MM.
- $\neg$CH routinely considered in the subject of cardinal characteristics of the continuum.

Philosophical attempts to settle CH

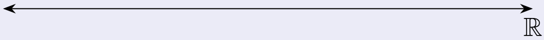
The truth or falsity of CH cannot be settled on the basis of proof from the ZFC axioms.

Set theorists have consequently offered various philosophical arguments aiming at a solution to the continuum problem, the problem of determining whether CH holds or its negation.

Freiling: throwing darts at the real line

The Axiom of Symmetry (Freiling JSL, 1986)

Asserts that for any function $x \mapsto A_x$ mapping reals to countable sets of reals, there are x, y with $y \notin A_x$ and $x \notin A_y$.



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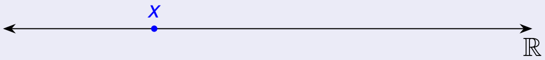

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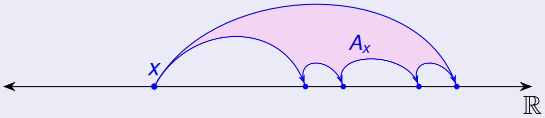

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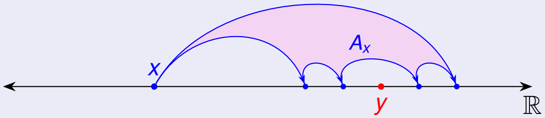

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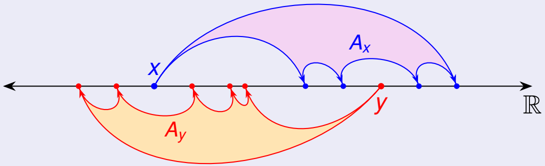

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Freiling's theorem was not generally accepted as a solution to CH, in light of non-measurable sets.

Similar attitudes toward Banach-Tarski in regard to AC.

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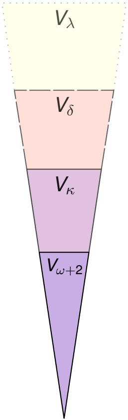
- Woodin made a case for $\neg$CH based on considerations of Ω-logic and forcing absoluteness. [Koe23]
- More recently, Woodin argues for CH based on features of his theory of Ultimate *L*, a canonical inner model accommodating even the largest large cardinals.

See [Rit15] for an account of Woodin's change of heart.

CH settled by second-order logic



Kreisel [Kre67] argues (also Isaacson [Isa11]) that CH is settled in second-order set theory ZFC_2.

Zermelo [Zer30] proved that the models of ZFC_2 are exactly V_κ for inaccessible cardinals κ. In particular, they agree a long way on the rank-initial segments V_α of the universe.

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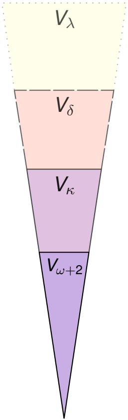

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Others criticize this observation as begging the question, or circular, since the very meaning of second-order logic is grounded in set theory.

CH at center of pluralism debate

The continuum hypothesis has often been at the center of the ongoing vigorous debate on pluralism taking place in the philosophy of set theory.

Universe view. According to this view, also known as *set-theoretic monism*, there is a unique absolute background concept of set, instantiated in the cumulative universe of all sets, in which set-theoretic assertions have a definite truth value.

On the universe view, every mathematical question comes to its final answer.

Main challenge for the universe view

The central discovery in set theory over the past half-century is the enormous range of set-theoretic possibility. The most powerful set-theoretical tools are most naturally understood as methods of constructing alternative set-theoretical universes, universes that seem fundamentally set-theoretic.

forcing, ultrapowers, canonical inner models, etc.

Much of set-theory research has been about constructing as many different models of set theory as possible. These models are often made to exhibit precise, exacting features or to exhibit specific relationships with other models.

Set-theoretic pluralism

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An analogy is often made to the nature of truth in geometry.

The dream solution

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I argue in [Ham15] that this is impossible.

> *Our situation with CH is not merely that CH is formally independent and we have no additional knowledge about whether it is true or not. Rather, we have an informed, deep understanding of how it could be that CH is true and how it could be that CH fails. We know how to build the CH and ¬CH worlds from one another. Set theorists today grew up in these worlds, comparing them and moving from one to another while controlling other subtle features about them. Consequently, if someone were to present a new set-theoretic principle Φ and prove that it implies ¬CH, say, then we could no longer look upon Φ as manifestly true for sets. To do so would negate our experience in the CH worlds, which we found to be perfectly set-theoretic. It would be like someone proposing a principle implying that only Brooklyn really exists, whereas we already know about Manhattan and the other boroughs. And similarly if Φ were to imply CH. We are simply too familiar with universes exhibiting both sides of CH for us ever to accept as a natural set-theoretic truth a principle that is false in some of them.*

Is CH an open question?

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Rather, the answer to CH consists of the deep body of knowledge that we have concerning how it behaves in the set-theoretic multiverse, how we can force it or its negation while preserving diverse other set-theoretic features.

How it might have been different

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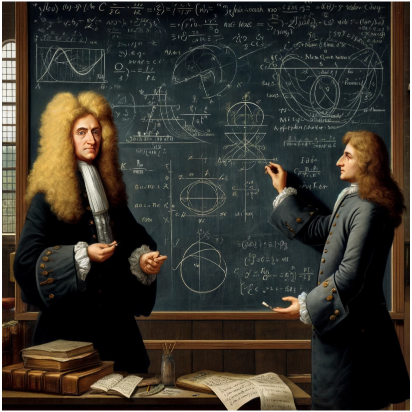
Namely, if our mathematical history had been just a little different, I claim, if certain mathematical discoveries had been made in a slightly different order, then we would naturally view the continuum hypothesis as a fundamental axiom of set theory, one furthermore necessary for mathematics and indeed indispensable for calculus.

The thought experiment



Let us imagine that in the early days of calculus, Newton and Leibniz had provided somewhat fuller accounts of their ideas about infinitesimals.

Thought experiment

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The infinitesimal foundations were mocked by Berkeley [Ber34]:

> *And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?*

It was simply not clear enough what kind of thing the infinitesimals were—were they part of the ordinary number system or did they somehow transcend it?

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Two “realms” of numbers:

- The real numbers $\mathbb{R}$
- A larger realm of numbers $\mathbb{R}^*$, let us call them *hyperreal*

This proposal immediately addresses the withering Berkeley criticism.

Clarifying nature of infinitesimals

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- Immediately releases tension of paradoxical claim that infinitesimals are positive, yet smaller than every positive number
- Enables us to clarify more precisely how the infinitesimals relate to the real numbers.
- Enables a frank discussion of the nature of the real numbers and infinitesimals

Infinitesimal existence

Let us imagine Leibniz clarifying the existence of infinitesimals.

Imaginary Leibniz

"Every conceivable gap in $\mathbb{R}$ *is filled by infinitesimals."*

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- Reciprocals of infinitesimals are infinite hyperreal numbers.
- Hyperreal field is not Archimedean.

Filling gaps

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Hyperreal field $\mathbb{R}^*$ is countably saturated

Every countable gap in $\mathbb{R}^*$

$$x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \qquad\qquad \cdots \leqslant y_2 \leqslant y_1 \leqslant y_0$$

with $x_i < y_i$ is filled by some hyperreal number z strictly between

$$x_0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \quad < z < \quad \cdots \leqslant y_2 \leqslant y_1 \leqslant y_0$$

Historical Leibniz

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These ideas led to further work of Hausdorff and later Hardy on the orders of infinity.

Two number systems, same laws and truths

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The hyperreal numbers thus fulfill the associativity and distributivity laws, and indeed any law that is true for the real numbers.

This same-laws view justifies the common calculations with infinitesimals that one finds in calculus.

Incipient transfer principle

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From contemporary perspective, this is an incipient form of:

Transfer principle

All structure on the real numbers $\mathbb{R}$ extends to the hyperreal numbers with the same truth.

$$\langle \mathbb{R}, +, \cdot, 0, 1, <, \mathbb{Z}, f, \ldots \rangle \prec \langle \mathbb{R}^*, +, \cdot, 0, 1, <, \mathbb{Z}^*, f^*, \ldots \rangle$$

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> *These Lemmas are premised to avoid the tediousness of deducing perplexed demonstrations ad absurdum, according to the method of the ancient geometers. [New46, p.102]*

According to Newton, the methods of infinitesimals and fluxions were eliminable, a conservative extension of standard mathematics, with nothing new.

A modest proposal

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The proposal is not that they somehow have a full-blown well-formulated theory of the hyperreal numbers.

Rather, further development and rigor will naturally come in time, just as it did in our actual mathematical history.

Fundamentally coherent account of infinitesimals

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See [Kei00] for an example of how one can develop the whole theory quite successfully on the basis of very primitive notions.

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Meanwhile, in the thought experiment, all the actual insights would be made and more, with increasing rigor and sophistication—an enduring calculus based on infinitesimals, proceeding roughly along the lines of nonstandard analysis.

Hyperreal numbers as a familiar number system

At bottom, the proposal is that the hyperreal numbers would have become one of the standard number systems that mathematicians discovered and became familiar with.

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We would find the hyperreal numbers alongside the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers, serving as one of the familiar standard number systems.

Gödel on infinitesimals

Kurt Gödel comes out in favor of the thought experiment history.

> *There are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.*
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> *Arithmetic starts with the integers and proceeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus.* [G9̈0, p. 311]

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Our actual mathematical history is odd and strange.

The thought experiment history is far more natural.

On the necessity of categoricity

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Thus, the categorical accounts of our particular structures become the framework of our mathematical reality.

Categoricity

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A *categorical* account of a mathematical structure identifies axioms true in that structure, such that those axioms furthermore determine that structure up to isomorphism.

Categorical accounts generally use second-order logic (impossible in first-order logic).

Categorical accounts of the central structures of mathematics

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Each of the central structures in mathematics enjoys a clear categorical characterization.

Importance of categoricity

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- A philosophical strand, growing out of Benacerraf.
- A mathematical strand, tracing to Dedekind categoricity.

Hyperreal categoricity

In the imaginary history, the hyperreal numbers $\mathbb{R}^*$ have become a core mathematical conception, present from the beginning at the foundations of calculus, and mathematicians would insist upon a categorical account.

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Is it possible? Can we have a categorical account the hyperreals $\mathbb{R}^*$?

The key event

Yes, we can give a categorical account of the hyperreal numbers. Let us imagine a Zermelo-like figure who proves:

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This can be proved in a back-and-forth argument, much like Cantor's DLO argument, but with transfinite length ω_1.

Sharp forms of hyperreal categoricity

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Theorem (Erdős, Gillman, and Henriksen [EGH55])

Under CH there is up to isomorphism only one real-closed field of size continuum whose order is countably saturated.

These hypotheses are close to the original principles we had attributed to our imaginary Newton and Leibniz.

In this way, from CH we can prove there is a unique hyperreal structure up to isomorphism.

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Meanwhile, there is no categoricity result without CH.

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In short,

- With CH, we have categoricity for the hyperreals.
- Without CH, we lack categoricity for the hyperreals.

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Lack of categoricity $\Longrightarrow$ lack of reference.

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Categoricity is required for a coherent structuralist practice.

How CH gets on the list of fundamental axioms

The thought experiment, at bottom

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- We know that CH is required for this.

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- Zermelo-like figure introduces ZFC $+ \mathrm{CH}$ to prove categoricity.
- We know that CH is required for this.

Thus, CH becomes necessary part of the foundational theory establishing the basic coherence of the hyperreal numbers.

CH becomes indispensable for the foundations of calculus.

Extrinsic justification of CH

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Similar to the current justification of ZFC in light of its successful account of the real numbers $\mathbb{R}$.

CH would be seen as vital for the account of the hyperreals $\mathbb{R}^*$, a core mathematical structure in the thought-experiment world.

Intrinsic justification

After the extrinsic justification, CH will naturally find intrinsic justification.

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For CH, it asserts agreement for the two known methods of achieving uncountability. $\aleph_1 = \beth_1$.

This is a unifying, explanatory principle of the uncountable, therefore intrinsic justification of CH.

Categoricity for GCH

A critic might want hyperreal fields of larger cardinalities.

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Natural continuation of saturation ideas from Leibniz, Euler to Hausdorff, Hardy and into contemporary times.

A vast elementary chain, converging to the surreal numbers, with its own class-theoretic categorical account.

Contingency of ZFC

Penelope Maddy [Mad88] on historical contingency of ZFC.

> *The fact that these few [ZFC] axioms are commonly enshrined in the opening pages of mathematics texts should be viewed as an historical accident, not a sign of their privileged epistemological or metaphysical status. [Mad88]*

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She was mainly concerned about large cardinal extensions of ZFC.

My thought experiment is another kind of historical contingency for ZFC, showing how we could naturally have taken CH as a basic principle, necessary for the success of mathematics.

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It would be perhaps like current attitudes about models of ZF with strange failures of the axiom of choice. For example, non-isomorphic algebraic closures of $\mathbb{Q}$. Often considered weird.

Similarly odd to have multiple non-isomorphic hyperreal fields, reinforcing view that ZFC $+ \mathrm{CH}$ is the right theory.

Different view of forcing

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Forcing would be seen as less successful, since CH not preserved.

In actual world, major attraction of forcing is that it preserves the fundamental theory ZFC. No longer true in imaginary world, since CH not preserved.

Forcing viewed like symmetric model construction—a means to produce weird counterexample models. Unnatural without CH.

Conclusion

We could have had a very different perspective on the continuum hypothesis.

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But a cateogorical account of $\mathbb{R}^*$ is possible only with CH.

So CH would have been a part of our foundational theory. Extrinsically justified, but then also intrinsically.

We would have viewed CH as necessary for mathematics, indispensible even for calculus.

Further reading

For more in this vein, see my recent paper:

[Ham24] Joel David Hamkins, "How the continuum hypothesis could have been a fundamental axiom," Journal for the Philosophy of Mathematics (2024)1:113–126, DOI:10.36253/jpm-2936.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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