From definiteness of objects to definiteness of truth

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame
VRF, Mathematical Institute, Oxford

NUS 2024
(In)determinacy in mathematics

This talk is based on joint work with Ruizhi Yang of Fudan University, Shanghai.

Our paper is available on the arXiv:

- [HY14] J. D. Hamkins, R. Yang, “Satisfaction is not absolute,” arxiv:1312.0670.

Arithmetic definiteness

Many mathematicians and philosophers regard the natural numbers $0, 1, 2, \ldots$, with their usual structure, as having a privileged, definite mathematical existence.

It is a part of the Platonic realm in which the number objects have a definite, absolute existence, and arithmetic assertions have definite, absolute truth values.

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

Arithmetic definiteness

Many mathematicians and philosophers regard the natural numbers $0, 1, 2, \ldots,$ with their usual structure, as having a privileged, definite mathematical existence.

It is a part of the Platonic realm in which the number objects have a definite, absolute existence, and arithmetic assertions have definite, absolute truth values.

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

Although some assertions we can neither prove nor refute over a given theory, nevertheless there is a fact of the matter about whether they are true.

I should like to tease apart these two kinds of definiteness, definiteness of objects versus definiteness of truth.

Definiteness of objects

For a structure $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ to be definite, what we would seem to mean is that

- The domain $\mathbb{N}$ of objects should be definite.
- Definite = determinate, clear, unambiguous, absolute, lacking contingency

Definiteness of objects

For a structure $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ to be definite, what we would seem to mean is that

- The domain $\mathbb{N}$ of objects should be definite.
- Definite = determinate, clear, unambiguous, absolute, lacking contingency
- Membership in the domain should be definite.
- The operations $x + y = z$, $r \cdot s = t$ should be definite.
- The relations $x < y$ should be definite.

In short, all the atomic structure is definite.

Definiteness of truth

For arithmetic truth to be definite, we would mean that for any arithmetic assertion φ, there is a definite fact of the matter about whether φ is true or not.

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

The view is that either there definitely are, or there definitely are not, infinitely many prime pairs. Riemann's hypothesis is either definitely true or definitely false—we just don't know which yet. The busy beaver function exhibits exactly the perfect, actual definitive values that it does.

Definiteness of objects $\to$ definiteness of truth?

Main Question

To what extent does definiteness about objects and structure lead to definiteness in truth?

Definiteness of objects $\to$ definiteness of truth?

Main Question

To what extent does definiteness about objects and structure lead to definiteness in truth?

Does having definiteness for objects and atomic structure entitle us to definiteness in the corresponding theory of truth?

Does a commitment to the definite nature of the natural numbers $0, 1, 2, \ldots$ and their atomic structure ensure also a definite theory of arithmetic truth?

From objects to truth

If the atomic structure is definite, it may seem reasonable in consequence that all truth assertions

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

would also be definite.

From objects to truth

If the atomic structure is definite, it may seem reasonable in consequence that all truth assertions

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

would also be definite. After all,

- Semantics seem to flow in a determinate recursive manner.

From objects to truth

If the atomic structure is definite, it may seem reasonable in consequence that all truth assertions

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

would also be definite. After all,

- Semantics seem to flow in a determinate recursive manner.
- Atomic facts are definite by assumption.
- Logical connectives are defined in a definite manner.
- Quantifiers range over a given definite domain of individuals.

From objects to truth

If the atomic structure is definite, it may seem reasonable in consequence that all truth assertions

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle \models \varphi$$

would also be definite. After all,

- Semantics seem to flow in a determinate recursive manner.
- Atomic facts are definite by assumption.
- Logical connectives are defined in a definite manner.
- Quantifiers range over a given definite domain of individuals.
- Truth conditions for any assertion thus flow from the nature of the objects themselves.

So it may seem reasonable to hold that definiteness of objects should lead to definiteness of truth.

Definiteness down low, indefiniteness up high

Nik Weaver, Solomon Feferman and others have defended a view whereby arithmetic truth has a definite character, while higher-order truth, such as set-theoretic assertions at the level of $P(\mathbb{N})$ and above, are less definite.

On such a view one might view the continuum hypothesis as a vague mathematical assertion, not capable of genuine resolution.

From objects to truth

Some philosophers seem to take the step from definiteness of objects to definiteness of truth.

Solomon Feferman (EFI 2013):

> In my view, the conception [of the bare structure of the natural numbers] is completely clear, and thence all arithmetical statements are definite.

It is Feferman's 'thence' to which I call attention.

From objects to truth

Some philosophers seem to take the step from definiteness of objects to definiteness of truth.

Solomon Feferman (EFI 2013):

> In my view, the conception [of the bare structure of the natural numbers] is completely clear, and thence all arithmetical statements are definite.

It is Feferman's 'thence' to which I call attention.

Donald Martin (EFI 2012):

> What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers.

An underlying mathematical question

Let us try to sharpen the philosophical dispute by formulating a purely mathematical version:

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

An underlying mathematical question

Let us try to sharpen the philosophical dispute by formulating a purely mathematical version:

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

This would seem to speak directly to the question whether one might regard objects as definite, while truth is contingent.

If the answer is no, this would be a mathematical sense in which definiteness of truth follows from definiteness of objects.

An underlying mathematical question

Let us try to sharpen the philosophical dispute by formulating a purely mathematical version:

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

This would seem to speak directly to the question whether one might regard objects as definite, while truth is contingent.

If the answer is no, this would be a mathematical sense in which definiteness of truth follows from definiteness of objects.

But if the answer is yes...

Indeterminateness in truth

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

Indeterminateness in truth

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The answer, we prove, is yes, even for key structures. [HY14]

- Models of set theory can have the same $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, yet disagree on arithmetic truth.

Indeterminateness in truth

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The answer, we prove, is yes, even for key structures. [HY14]

- Models of set theory can have the same $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, yet disagree on arithmetic truth.
- Models of set theory can have the same reals, yet disagree on projective truth.

Indeterminateness in truth

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The answer, we prove, is yes, even for key structures. [HY14]

- Models of set theory can have the same $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$, yet disagree on arithmetic truth.
- Models of set theory can have the same reals, yet disagree on projective truth.
- Models of set theory can have V_δ in common, yet disagree about whether it is a model of ZFC.
- and many more...

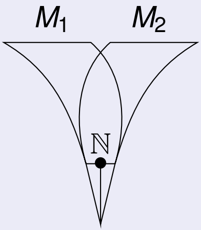
These are instances in which we can seem to have definiteness of objects without definiteness of truth.

Theorem

If ZFC *is consistent, then there are models* $M_1, M_2 \models$ ZFC *which have the same arithmetic structure*

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



There is a sentence σ *in* M_1 *and* M_2

M_1 *believes* $\mathbb{N} \models \sigma$

M_2 *believes* $\mathbb{N} \models \neg\sigma$

Standard models

We all know *the* standard model of arithmetic: $\langle\mathbb{N},+,\cdot,0,1,<\rangle$.

A model of arithmetic is *a* standard model, or ZFC-*standard*, if it is the standard model of arithmetic $\langle\mathbb{N},+,\cdot,0,1,<\rangle^M$ extracted from some model of set theory $M\models$ ZFC.

So a standard model of arithmetic is precisely one that is thought to be the standard model of arithmetic from the perspective of some model of set theory.

If $M\models$ ZFC, we may also extract its version of the theory of true arithmetic, $\mathrm{TA}^M=\{\,\sigma\mid M\models\mathbb{N}\models\sigma\,\}$.

Satisfaction class

A *truth predicate* on a model of arithmetic $\mathcal{N} = \langle N, +, \cdot, 0, 1, < \rangle$ is a class $\mathrm{Tr} \subseteq N$ satisfying the Tarskian recursion:

- For σ atomic, $\sigma \in \mathrm{Tr}$ just in case $\mathcal{N}$ thinks σ is true.
- $\sigma \wedge \tau \in \mathrm{Tr}$ iff $\sigma \in \mathrm{Tr}$ and $\tau \in \mathrm{Tr}$.
- $\neg\sigma \in \mathrm{Tr}$ if $\sigma \notin \mathrm{Tr}$.
- $\exists x\, \varphi(x) \in \mathrm{Tr}$ iff $\varphi(\underbrace{1 + \cdots + 1}_{n}) \in \mathrm{Tr}$ some $n \in N$.

Tarski: No such truth predicate is definable in $\mathcal{N}$.

Every ZFC-standard model of arithmetic has an *inductive* truth predicate, meaning $\langle N, +, \cdot, 0, 1, <, \mathrm{Tr} \rangle \models \mathrm{PA}(\mathrm{Tr})$.

Incompatible truth predicates

Theorem (Krajewski 1974)

There are models of arithmetic with different incompatible inductive truth predicates.

Proof.

Suppose $\mathcal{N}_0 = \langle N_0, +, \cdot, 0, 1, < \rangle$ has an inductive truth predicate. Let T be the elementary diagram $\Delta(\mathcal{N}_0)$, plus "Tr is an inductive truth predicate." This is consistent. Any model of T provides an elementary extension of $\mathcal{N}_0$. If they all have a unique truth predicate, then Tr would be implicitly definable in the sense of Beth's implicit definability theorem, and hence by Beth must be explicitly definable, which contradicts Tarski's theorem. So there is an elementary extension $\mathcal{N}$ of $\mathcal{N}_0$ with at least two different truth predicates. □

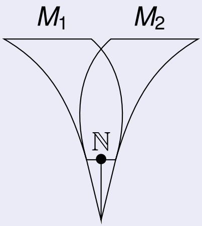
Satisfaction is not absolute

Theorem

If ZFC *is consistent, then there are* $M_1, M_2 \models$ ZFC *which have the same natural numbers and arithmetic structure*

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



There is a sentence σ *in* M_1 *and* M_2

M_1 *believes* $\mathbb{N} \models \sigma$

M_2 *believes* $\mathbb{N} \models \neg\sigma$

Same objects, different truth.

Proof

Fix any countable $M_1 \models$ ZFC with $\langle \mathbb{N}, \mathrm{TA} \rangle^{M_1}$ computably saturated.

Claim there are sentences σ, τ with same 1-type in $\mathbb{N}^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1}$, but M_1 thinks σ is true and τ false in $\mathbb{N}^{M_1}$. (Proof: consider the type $p(s,t)$ containing $\varphi(s) \leftrightarrow \varphi(t)$ and $s \in \mathrm{TA}$ and $t \notin \mathrm{TA}$; this is finitely realized, since TA is not definable.)

By back-and-forth, there is automorphism $\pi : \mathbb{N}^{M_1} \to \mathbb{N}^{M_1}$ with $\pi(\tau) = \sigma$.

Build a copy M_2 of M_1 so that π extends to an isomorphism $\pi^* : M_1 \to M_2$. So $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. But M_1 thinks σ is true, yet M_2 thinks $\sigma = \pi^*(\tau)$ is false. $\square$

A generalization

Theorem

For any countable $M \models \mathrm{ZFC}$, any structure $\mathcal{N} \in M$ finite language, any $S \subseteq N$ in M not definable in $\mathcal{N}$. Then there are $M \prec M_1$ and $M \prec M_2$ with $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$, yet $S^{M_1} \neq S^{M_2}$.

Note S^{M_1} and S^{M_2} share all properties of S in M.

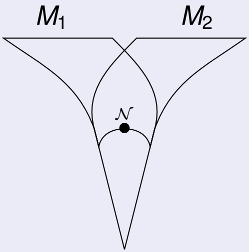
Proof.

Fix $M \prec M_1$ countable computably saturated. So $\langle \mathcal{N}, S \rangle^{M_1}$ is computably saturated. Since S not definable, there are $a, b \in \mathcal{N}^{M_1}$ with same 1-type in $\mathcal{N}^{M_1}$, but $a \in S, b \notin S$. So $\exists \pi : \mathcal{N}^{M_1} \cong \mathcal{N}^{M_1}$ with $\pi(b) = a$. Extend π to $\pi^* : M_1 \cong M_2$. So $a \in S^{M_1}$ but $a = \pi(b) \notin S^{M_2}$. □

Satisfaction is not absolute

Corollary

If M is a countable model of set theory and $\mathcal{N}$ is (sufficiently robust) structure in M, in a finite language. Then there are $M \prec M_1$ and $M \prec M_2$, which agree on the natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and on $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$, yet disagree on satisfaction $\mathcal{N} \models \sigma[\vec{a}]$ for this structure.



$M \prec M_1, M_2 \models \text{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2}, \qquad \mathcal{N}^{M_1} = \mathcal{N}^{M_2}$

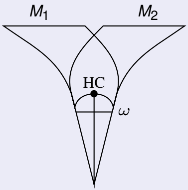
there are σ and $\vec{a}$ for which

M_1 *believes* $\mathcal{N} \models \sigma[\vec{a}]$

M_2 *believes* $\mathcal{N} \models \neg\sigma[\vec{a}]$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, their reals $\mathbb{R}^{M_1} = \mathbb{R}^{M_2}$ and their hereditarily countable sets $\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$, but which disagree on their theories of projective truth.



$M_1, M_2 \models \mathrm{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2} \qquad \mathbb{R}^{M_1} = \mathbb{R}^{M_2}$

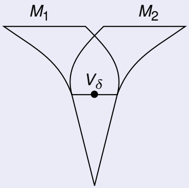
$\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$

M_1 *believes* $\mathrm{HC} \models \sigma$

M_2 *believes* $\mathrm{HC} \models \neg\sigma$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which have a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, but which disagree on truth in this structure.



$M_1, M_2 \models \mathrm{ZFC}$

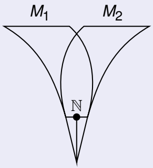
$V_\delta^{M_1} = V_\delta^{M_2} \models \mathrm{ZFC}$

M_1 *believes* $V_\delta \models \sigma$

M_2 *believes* $V_\delta \models \neg\sigma$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers with successor, addition and order $\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$, but which disagree on natural-number multiplication, so that M_1 thinks $a \cdot b = c$ for some particular natural numbers, but M_2 disagrees.



$M_1, M_2 \models \text{ZFC}$

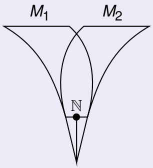
$\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$

M_1 *believes* $\mathbb{N} \models a \cdot b = c$

M_2 *believes* $\mathbb{N} \models a \cdot b \neq c$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers with successor and order $\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$, but which disagree on the even numbers, the prime numbers and the powers of two, so that M_1 thinks some n is a large odd prime number, but M_2 thinks it is a large power of 2.



$M_1, M_2 \models \text{ZFC}$

$\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$

M_1 *believes* $\mathbb{N} \models$ *n is an odd prime*

M_2 *believes* $\mathbb{N} \models n = 2^k$ *for some k*

Iterated truth predicates

Begin with the standard model $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Add a truth predicate $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0 \rangle$, where Tr_0 is a truth predicate for arithmetic assertions.

Add a truth predicate for that structure, $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle$, where Tr_1 is a truth predicate for assertions in the language with Tr_0.

And so on $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$...

Truth about truth is not absolute

Corollary

For every countable model of set theory M and any natural number n, there are $M \prec M_1$ *and* $M \prec M_2$ *with* $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ *and same iterated truths up to n*

$$\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_2}$$

but which disagree on the next order of truth Tr_{n+1}*.*

The point: Tr_{n+1} is not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$.

Disagreement on the Church-Kleene ordinal

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and have a computable linear order $\lhd$ on $\mathbb{N}$ in common, yet M_1 thinks $\langle \mathbb{N}, \lhd \rangle$ is a well-order and M_2 does not.

Proof.

Being the computable index of a well-order is Π^1_1-complete and hence not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. □

Disagreement on definability

Theorem

Every countable model of set theory M has $M \prec M_1$ and $M \prec M_2$, which agree on

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$$

and which have a set $A \subseteq \mathbb{N}$ in common, yet M_1 thinks A is first-order definable in $\mathbb{N}$ and M_2 thinks it is not.

The proof relies on the non-absoluteness theorem, plus:

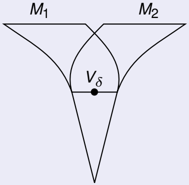
Lemma (Andrew Marks)

There is $B \subseteq \mathbb{N} \times \mathbb{N}$, such that $\{\, n \in \mathbb{N} \mid B_n \text{ is arithmetic} \,\}$ is not definable in the structure $\langle \mathbb{N}, +, \cdot, 0, 1, <, B \rangle$.

Precise violation of ZFC

Theorem

Every countable model of set theory $M \models$ ZFC has elementary extensions M_1 and M_2, with a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, such that M_1 thinks that the least natural number n for which V_δ violates Σ_n-collection is even, but M_2 thinks it is odd.



$M_1, M_2 \models$ ZFC

$V_\delta^{M_1} = V_\delta^{M_2}$

n is least with $\neg\Sigma_n$-collection in V_δ

M_1 believes n is even

M_2 believes n is odd

Proof

Suppose that $M \models \text{ZFC}$, and let

$$T_1 = \Delta(M) + \ V_\delta \prec V \ + \ \{\, m \in V_\delta \mid m \in M \,\}$$
$$+ \ \text{the least } n \text{ such that } V_\delta \not\models \Sigma_n\text{-collection is even},$$

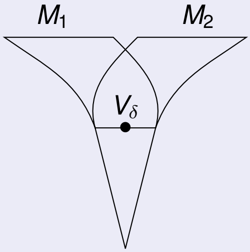
Consistent via the reflection theorem. Similar with theory T_2, where we assert n is odd.

Let $\langle M_1, M_2 \rangle$ be a computably saturated model pair, with $M_1 \models T_1$ and $M_2 \models T_2$. It follows that $\langle V_\delta^{M_1}, V_\delta^{M_2} \rangle$ is a computably saturated model pair of elementary extensions of $\langle M, \in^M \rangle$, which are therefore elementarily equivalent in the language of set theory with constants for elements of M, and hence isomorphic by an isomorphism respecting those constants. So without loss $\langle M, \in^M \rangle \prec \langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$. Meanwhile, M_1 thinks that this V_δ violates Σ_n-collection first at an even n and M_2 thinks it does so first for an odd n, as desired. $\square$

Disagreement about whether $V_\delta \models \mathrm{ZFC}$

Theorem

If M is a countable model of set theory in which the worldly cardinals form a stationary proper class, then there are $M \prec M_1$ and $M \prec M_2$ with $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$, but M_1 thinks $V_\delta \models \mathrm{ZFC}$ and M_2 thinks $V_\delta \not\models \mathrm{ZFC}$.



$M_1, M_2 \models \mathrm{ZFC}$

$\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$

M_1 *believes* $V_\delta \models \mathrm{ZFC}$

M_2 *believes* $V_\delta \not\models \mathrm{ZFC}$

Proof

Fix any countable $M \models \text{ZFC}$, with worldly cardinals a stationary proper class. Let $T_1 = \Delta(M) + V_\delta \prec V$, plus $a \in V_\delta$ for each $a \in M$, plus "δ is worldly." Every finite subtheory is consistent, because for any standard n there is a club of Σ_n-correct cardinals, and so one of them is worldly in M. So T_1 is consistent.

Similarly, let T_2 assert the same, except instead "δ is not worldly." This theory is also finitely consistent and hence consistent.

Let $\langle M_1, M_2 \rangle$ be a computably saturated model pair, where $M_1 \models T_1$ and $M_2 \models T_2$. So $\langle V_\delta^{M_1}, V_\delta^{M_2} \rangle$ is computably saturated model pair, both with the diagram of M. So they are isomorphic, preserving M. So we may assume $\langle M, \in^M \rangle \prec \langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$. The theories T_1 and T_2 ensure that $V_\delta \prec V$ in both M_1 and M_2, and that $M_1 \models \delta$ *is worldly* and $M_2 \models \delta$ *is not worldly*, or in other words, $M_1 \models (V_\delta \models \text{ZFC})$, but $M_2 \models (V_\delta \not\models \text{ZFC})$, as desired.

Nonstandardness objection

Objection

The indeterminant sentence σ is necessarily nonstandard.

Nonstandardness objection

Objection

The indeterminant sentence σ is necessarily nonstandard.

Several responses

- The sentence σ has Gödel code inside the given structures $\ulcorner\sigma\urcorner \in \mathcal{N}^{M_1} = \mathbb{N}^{M_2}$.
- The sentence σ is standard inside M_1 and M_2, with $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. Both models M_1 and M_2 think σ is perfectly legitimate.

Nonstandardness objection

Objection

The indeterminant sentence σ is necessarily nonstandard.

Several responses

- The sentence σ has Gödel code inside the given structures $\ulcorner\sigma\urcorner \in \mathcal{N}^{M_1} = \mathbb{N}^{M_2}$.
- The sentence σ is standard inside M_1 and M_2, with $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. Both models M_1 and M_2 think σ is perfectly legitimate.
- M_1 and M_2 amount to different, inequivalent metatheoretic contexts to consider same arithmetic structure $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$.

Nonstandardness objection

Objection

The indeterminant sentence σ is necessarily nonstandard.

Several responses

- The sentence σ has Gödel code inside the given structures $\ulcorner\sigma\urcorner \in \mathcal{N}^{M_1} = \mathbb{N}^{M_2}$.
- The sentence σ is standard inside M_1 and M_2, with $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. Both models M_1 and M_2 think σ is perfectly legitimate.
- M_1 and M_2 amount to different, inequivalent metatheoretic contexts to consider same arithmetic structure $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$.
- We pull apart definiteness for particular assertions versus universal claim that all sentences are definite.
- Theory of truth is higher-order realm, applying to all sentences available.

Monism versus pluralism

Arithmetic monism (arithmetic universe view)

This is the view that there is an intended model of arithmetic, a unique definite arithmetic structure

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle,$$

consisting of the numbers 0, 1, 2, and so on, which furthermore carries a definite theory of arithmetic truth.

Monism versus pluralism

Arithmetic monism (arithmetic universe view)

This is the view that there is an intended model of arithmetic, a unique definite arithmetic structure

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle,$$

consisting of the numbers 0, 1, 2, and so on, which furthermore carries a definite theory of arithmetic truth.

Arithmetic pluralism

This is the view, in contrast, that we are mistaken in our ideas about an absolute standard conceptions of arithmetic—ultimately there are diverse incompatible, incomparable conceptions of arithmetic.

Objectors

Objection

Joel, are you seriously telling us that you don't believe in the definiteness of arithmetic?

Objectors

Objection

Joel, are you seriously telling us that you don't believe in the definiteness of arithmetic?

You know, 0, 1, 2, 3, and so on.

Objectors

Objection

Joel, are you seriously telling us that you don't believe in the definiteness of arithmetic?

You know, 0, 1, 2, 3, and so on.

Response

My view is based in the worry that we are making a mistake in presuming we have a definite concept of the finite. The arguments for definiteness are actually quite weak.

Objectors

Objection

Joel, are you seriously telling us that you don't believe in the definiteness of arithmetic?

You know, 0, 1, 2, 3, and so on.

Response

My view is based in the worry that we are making a mistake in presuming we have a definite concept of the finite. The arguments for definiteness are actually quite weak.

What are the best arguments for arithmetic definiteness?

Clear and definite conception

People claim to have a clear and definite conception.

The natural numbers are 0, 1, 2, and so on.

Clear and definite conception

People claim to have a clear and definite conception.

The natural numbers are 0, 1, 2, and so on.

But what is this “and so on”?

Does it express a clear enough notion?

What are the finite numbers?

The idea is that the numbers start with 0, and we systematically find a successor number for every number that we have so far.

The natural numbers are all the numbers that are produced in this process.

How many times are we to apply the successor operation?

What are the finite numbers?

The idea is that the numbers start with 0, and we systematically find a successor number for every number that we have so far.

The natural numbers are all the numbers that are produced in this process.

How many times are we to apply the successor operation?

Finitely many times, of course.

What are the finite numbers?

The idea is that the numbers start with 0, and we systematically find a successor number for every number that we have so far.

The natural numbers are all the numbers that are produced in this process.

How many times are we to apply the successor operation?

Finitely many times, of course.

But is this circular? We are trying to define the notion of finite.

Several methods of solving this have been proposed.

Fregean anscestral

Frege defines the finite numbers as those numbers that have every property of 0 that is passed from every number to its successor.

Fregean anscestral

Frege defines the finite numbers as those numbers that have every property of 0 that is passed from every number to its successor.

So the concept of finiteness makes use of:

- independent concept of “number”
- concept of “successor” relation on numbers
- concept of “concept” holding of a number

Fregean anscestral

Frege defines the finite numbers as those numbers that have every property of 0 that is passed from every number to its successor.

So the concept of finiteness makes use of:

- independent concept of “number”
- concept of “successor” relation on numbers
- concept of “concept” holding of a number

Are these prior notions clear and definite enough to help us establish the definiteness of our concept of the finite?

Fregean anscestral

Frege defines the finite numbers as those numbers that have every property of 0 that is passed from every number to its successor.

So the concept of finiteness makes use of:

- independent concept of “number”
- concept of “successor” relation on numbers
- concept of “concept” holding of a number

Are these prior notions clear and definite enough to help us establish the definiteness of our concept of the finite?

To establish the definiteness of finiteness, after all, we would seem to need first to establish the definiteness of *those* notions as well.

Dedekind categoricity

Mathematicians often mention the categorical characterization of the finite numbers $\langle \mathbb{N}, 0, S \rangle$ provided by Dedekind's theory of successor.

Dedekind categoricity

Mathematicians often mention the categorical characterization of the finite numbers $\langle \mathbb{N}, 0, S \rangle$ provided by Dedekind's theory of successor.

- 0 is not a successor
- Successor operation $S : \mathbb{N} \to \mathbb{N}$ is one-to-one.
- Every number is generated from 0 by successor. I.e. every set X containing 0 and closed under successor contains all the numbers.

Dedekind categoricity

Mathematicians often mention the categorical characterization of the finite numbers $\langle \mathbb{N}, 0, S \rangle$ provided by Dedekind's theory of successor.

- 0 is not a successor
- Successor operation $S : \mathbb{N} \to \mathbb{N}$ is one-to-one.
- Every number is generated from 0 by successor. I.e. every set X containing 0 and closed under successor contains all the numbers.

Dedekind proved that any two models of this theory are isomorphic. This theory determines a unique structure, the structure of finite numbers.

Dedekind categoricity

Mathematicians often mention the categorical characterization of the finite numbers $\langle \mathbb{N}, 0, S \rangle$ provided by Dedekind's theory of successor.

- 0 is not a successor
- Successor operation $S : \mathbb{N} \to \mathbb{N}$ is one-to-one.
- Every number is generated from 0 by successor. I.e. every set X containing 0 and closed under successor contains all the numbers.

Dedekind proved that any two models of this theory are isomorphic. This theory determines a unique structure, the structure of finite numbers.

This celebrated categoricity result leads eventually to the mathematical thread of the philosophy of structuralism.

My objection

Dedekind defines the concept of finite by reference to the higher-order concept of arbitrary set.

My objection

Dedekind defines the concept of finite by reference to the higher-order concept of arbitrary set.

Similarly, Frege defines the concept of finite by reference to a higher-order notion of “concept”.

My objection

Dedekind defines the concept of finite by reference to the higher-order concept of arbitrary set.

Similarly, Frege defines the concept of finite by reference to a higher-order notion of “concept”.

But how can we secure definiteness for the concept of finite by appealing to the comparatively murky realm of set theory and higher order logic?

Is the domain of all concepts more clear and definite than our prior notion of finiteness? Frankly, it seems much less clear and definite.

My objection

Dedekind defines the concept of finite by reference to the higher-order concept of arbitrary set.

Similarly, Frege defines the concept of finite by reference to a higher-order notion of "concept".

But how can we secure definiteness for the concept of finite by appealing to the comparatively murky realm of set theory and higher order logic?

Is the domain of all concepts more clear and definite than our prior notion of finiteness? Frankly, it seems much less clear and definite.

This strikes me as hopeless.

In the end we don't seem to have any good arguments for the definiteness of our concept of the finite. And the results about M_1 and M_2 show how it could be that there is nonabsoluteness in the higher-order notions.

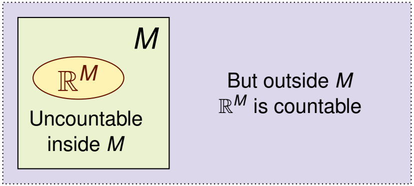
Skolem paradox

Meanwhile, we already understand quite deeply how the notions of countability and even finiteness can be nonabsolute.

Skolem paradox

Meanwhile, we already understand quite deeply how the notions of countability and even finiteness can be nonabsolute.

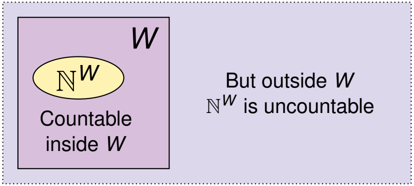
First, the Skolem paradox shows us, via the Löwenheim-Skolem theorem, that if there is any model of ZFC set theory at all, then there is a countable model $M \models \text{ZFC}$.



The paradox arrives because ZFC proves that there are uncountable sets, so there will be sets inside M that M believes uncountable, but they are actually countable.

Countable in a world, but actually uncountable

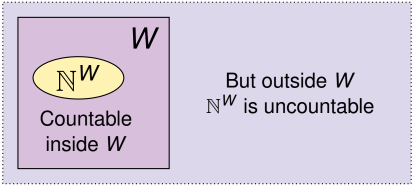
A converse nonabsoluteness: Countable inside W, but actually uncountable.



Write down theory ZFC, plus assertions "c_i is a natural number" and $c_i \neq c_j$ for uncountably many new constant symbols c_i.

Countable in a world, but actually uncountable

A converse nonabsoluteness: Countable inside W, but actually uncountable.

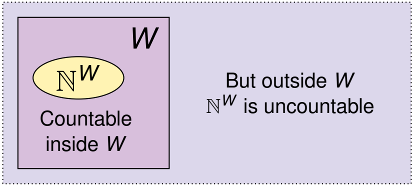


Write down theory ZFC, plus assertions "c_i is a natural number" and $c_i \neq c_j$ for uncountably many new constant symbols c_i.

Finitely consistent, hence consistent. So it has a model W. The $\mathbb{N}^W$ is thought countable inside W, but is actually uncountable.

Countable in a world, but actually uncountable

A converse nonabsoluteness: Countable inside W, but actually uncountable.



Write down theory ZFC, plus assertions "c_i is a natural number" and $c_i \neq c_j$ for uncountably many new constant symbols c_i.

Finitely consistent, hence consistent. So it has a model W. The $\mathbb{N}^W$ is thought countable inside W, but is actually uncountable.

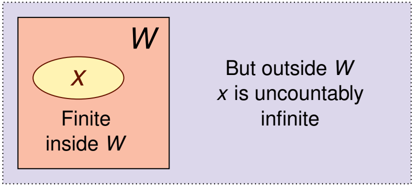
Alternative proof: ultrapower $W = V^{\mathbb{N}}/\mu$.

Finite in a world, but actually uncountably infinite

More extreme: finiteness is also nonabsolute!

Finite in a world, but actually uncountably infinite

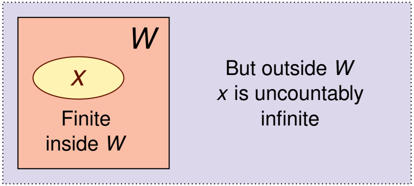
More extreme: finiteness is also nonabsolute!



There is a model W with x that is finite inside W, but actually uncountable outside W.

Finite in a world, but actually uncountably infinite

More extreme: finiteness is also nonabsolute!

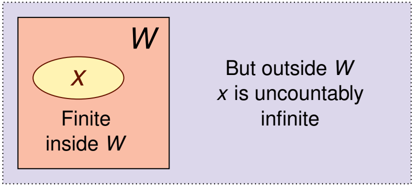


There is a model W with x that is finite inside W, but actually uncountable outside W.

Follows by same methods as earlier. Just take all c_i to be less than c, also asserted to be a natural number.

Finite in a world, but actually uncountably infinite

More extreme: finiteness is also nonabsolute!



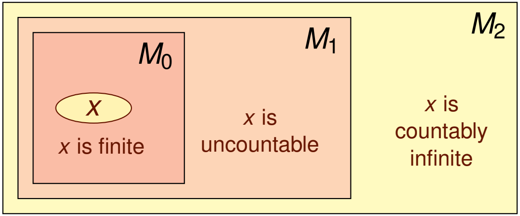
There is a model W with x that is finite inside W, but actually uncountable outside W.

Follows by same methods as earlier. Just take all c_i to be less than c, also asserted to be a natural number.

Also true in ultrapowers. Predecessors of a nonstandard number are uncountable outside the model.

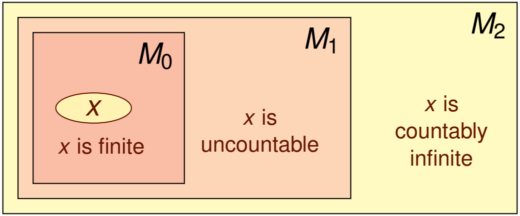
Iterative nonabsoluteness

A more elaborate display of nonabsoluteness.



Iterative nonabsoluteness

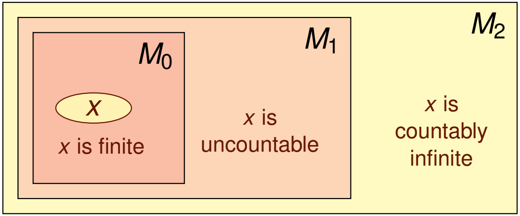
A more elaborate display of nonabsoluteness.



M_1 is any countable model of set theory; M_0 is an internal ultrapower of M_1.

Iterative nonabsoluteness

A more elaborate display of nonabsoluteness.

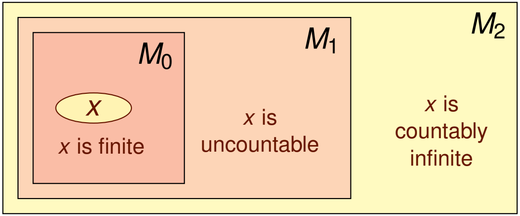


M_1 is any countable model of set theory; M_0 is an internal ultrapower of M_1.

Let x be predecessors of a nonstandard natural number in M_0. So x is finite in M_0, uncountable in M_1.

Iterative nonabsoluteness

A more elaborate display of nonabsoluteness.



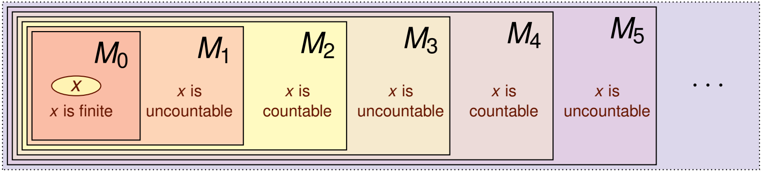
M_1 is any countable model of set theory; M_0 is an internal ultrapower of M_1.

Let x be predecessors of a nonstandard natural number in M_0. So x is finite in M_0, uncountable in M_1.

Let $M_2 = M_1[G]$ be forcing extension making x countable.

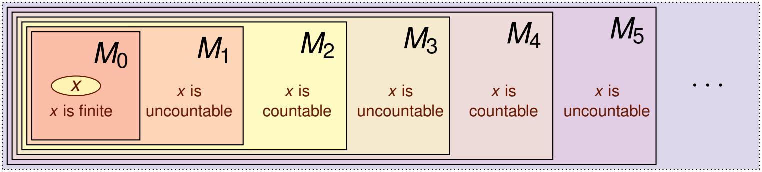
Arbitrary iterations

Can iterate that idea to make arbitrary length patterns.



Arbitrary iterations

Can iterate that idea to make arbitrary length patterns.



Start with finite, then alternate uncountable-countable as many times as desired.

Pseudo-countability

A structure x is *psuedo-countable* if there is a set theoretic world W in which x, with the same elements, is seen as countable.

Pseudo-countability

A structure x is *psuedo-countable* if there is a set theoretic world W in which x, with the same elements, is seen as countable.

This includes many uncountable structures.

Pseudo-countability

A structure x is *psuedo-countable* if there is a set theoretic world W in which x, with the same elements, is seen as countable.

This includes many uncountable structures.

In [Ham22] I use this concept to extend methods from the countable realm to uncountable instances.

Pseudo-countability

A structure x is *psuedo-countable* if there is a set theoretic world W in which x, with the same elements, is seen as countable.

This includes many uncountable structures.

In [Ham22] I use this concept to extend methods from the countable realm to uncountable instances.

In this way, the nonabsoluteness phenomenon, taken seriously, can be used for a mathematical purpose and lead to mathematical insight.

Definiteness of truth

The question was whether we may infer definiteness in our theory of mathematical truth as a consequence of the definiteness of our mathematical objects?

Many mathematicians and mathematical philosophers appear to do so.

Meanwhile, the mathematical results may appear to undermine that conclusion.

Perhaps our world is like M_1, where the natural numbers $\mathbb{N}^{M_1}$ are definite, yet have different truths in another world M_2.

Definiteness of truth is an additional commitment

My thesis is that the definiteness of the theory of truth for a structure does not follow as a consequence of the definiteness the structure in which that truth resides.

Rather, it must be seen as an additional and higher-order commitment to definiteness.

Arithmetic pluralism and indefiniteness

My view of the situation with arithmetic pluralism.

- Arithmetic pluralism is a natural default view in light of the general weakness of the arguments for monism.

Arithmetic pluralism and indefiniteness

My view of the situation with arithmetic pluralism.

- Arithmetic pluralism is a natural default view in light of the general weakness of the arguments for monism.
- The model-theoretic results paint a robust, satisfying picture of what it would be like for arithmetic pluralism to hold.
- Arithmetic pluralism brings down to the arithmetic realm a view that many find tractable in the higher realm.

Arithmetic pluralism and indefiniteness

My view of the situation with arithmetic pluralism.

- Arithmetic pluralism is a natural default view in light of the general weakness of the arguments for monism.
- The model-theoretic results paint a robust, satisfying picture of what it would be like for arithmetic pluralism to hold.
- Arithmetic pluralism brings down to the arithmetic realm a view that many find tractable in the higher realm.
- Arithmetic pluralism provides a fundamental explanation of the ubiquitous independence phenomenon.
- Arithmetic pluralism is more fully consonant with the idea that higher-order theories are genuinely new commitments that might have been otherwise.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
City University of New York

References I

[Ham22] Joel David Hamkins. “Pseudo-countable models”. *Mathematics arXiv* (2022). arXiv:2210.04838[math.LO]. http://jdh.hamkins.org/pseudo-countable-models.

[HY14] Joel David Hamkins and Ruizhi Yang. “Satisfaction is not absolute”. (2014), pp. 1–34. arXiv:1312.0670[math.LO]. http://wp.me/p5M0LV-Gf.