Did Turing ever halt?

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[HN24] Joel David Hamkins and Theodor Nenu, “Did Turing prove the undecidability of the halting problem?”, 18 pages, 2024, Mathematics arXiv:2407.00680. Under review.

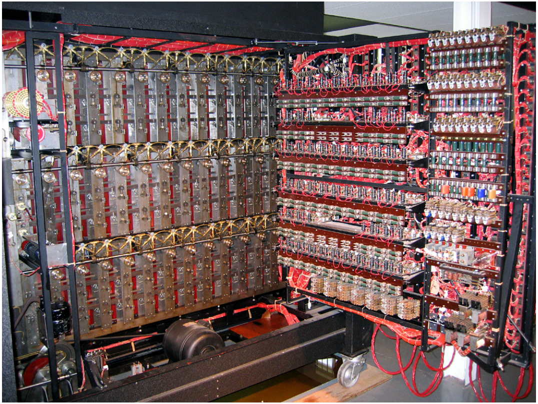
Alan Turing (1912–1954)

Sculpture by Stephen Kettle, Bletchley Park

Enigma

German ‘Enigma’ encryption device

The Bombe



Bombe device, Bletchley Park

On computable numbers, 1936

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173–198.

Turing's 1936 paper, "On computable numbers..."

It is difficult to overstate the importance of Turing's paper, written while he was a student at Cambridge.

Introduces profound, fundamental ideas on computability.

- Introduces Turing's formal conception of computability
- Defines Turing machines
- Proves existence of universal computers
- Identifies undecidability phenomenon
- Solves the Entscheidungsproblem
- Introduces the computable numbers

Turing laid the theoretical foundation for the computer age. One of the most impactful papers ever written.

Turing's concept of computability

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But of course, such a "computer" is not a machine, but a person—more specifically, an occupation. The computer rooms were filled with people hired as computers and tasked with various computational duties, often in finance or engineering.

In old photos, you can see the computers—mostly women—sitting at big wooden desks, with pencils and a sufficient supply of paper. They would perform their computations by writing on the paper, of course, according to various definite computational procedures.

What ‘computers’ do

Turing aimed to model an idealized form of computational processes.

He reflected on the behavior of ‘computers’ (that is, people working as computers) when working on a computational task. They make various marks on paper, depending in part on what they see before them, or after looking at earlier marks.

Eventually, some marks may be indicated as output information.

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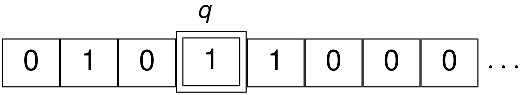
- May assume each cell has only 0 or 1.
- Two dimensions not important; assume a line of cells.
- May assume computer has limited memory, but as much paper as needed.
- Rudimentary actions determined by current ‘state’ of mind.

Turing machines

Thus, Turing derived his machine concept of computation.



A Turing machine has an infinite paper tape, divided into cells, which accommodate 0 and 1.

The head, at any moment in one of finitely many *states*, reads and writes on the tape, moving according to the rigid instructions of a program.

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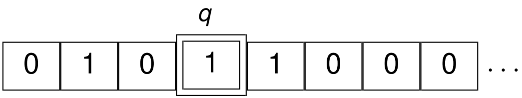

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$$(q, a) \mapsto (r, b, d).$$

When in state q reading symbol a, then change to state r, write symbol b, and move one cell in direction d.

Computable numbers

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He defines that a real number is *computable*, when there is a computational procedure to enumerate the decimal digits of the number.

$$3.14159265358979323846264338327950288419\cdots$$

He proceeds to develop the theory of computable real numbers.

Arithmetic

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$$\begin{array}{rl} & 0.111111111111\ldots \\ + & 0.222222222222\ldots \\ \hline & 0.333333333333\ldots \end{array}$$

We would like to compute the sum of two given numbers.

Arithmetic is more difficult than you expect!

But hang on. Consider the following case of $a + b$.

$$\begin{array}{rl} & 0.333333333333\ldots \\ + & 0.666666666666\ldots \\ \hline & 0.999999999999\ldots \end{array}$$

We can start writing down the answer 0.99999 . . ., but these digits will be wrong if we ever find a carry term, since then the answer should be 1.000000 . . .

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It seems that we cannot start to give the digits of $a + b$, given the initial digits of a and b.

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But if one modifies Turing's concept slightly, however, then everything works great!

A solution

Namely, logicians today define that a *computable real number* is a program that computes rational approximations to a real number, as accurately as desired.

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With this modified concept, all the usual operations are computable.

$$a + b \qquad ab \qquad \sin(x) \qquad e^x$$

Turing's computable real number idea turns into the robust subject known as *computable analysis*.

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It turns out, using extensions of Turing's arguments, that they are different.

Puzzling example

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...but when shall we say No?

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So in any case the problem is computable. We just don't know which algorithm works.

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Turing sought to answer these questions.

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But can we also compute the No answers?

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A beautiful self-referential argument

Suppose toward contradiction that we could solve the halting problem. Consider this strange algorithm q: On input p, a program, we ask whether program p would halt if given p itself as input; if yes, then we jump into an infinite loop; if no, then we halt immediately.

Let us run q on input q. It will halt if and only if it doesn't halt. Contradiction.

So the halting problem is undecidable.

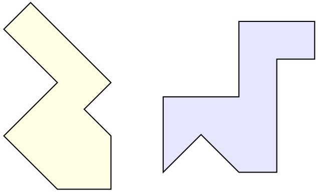
Undecidability

The undecidability phenomenon is now known to be pervasive in mathematics.

We have thousands of concrete decision problems, which cannot in principle be solved by any computational procedure.

Often, one can prove that a problem is undecidable by embedding the halting problem inside it.

The tiling problem



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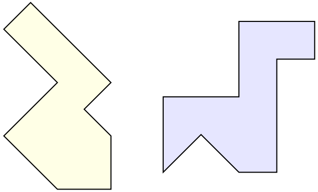

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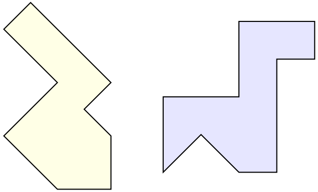

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Answer. No. There is no computational procedure that correctly solves this problem.

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- Elementary expression. Decide if a given functional expression in x using rationals, π, $\ln 2$, $+, -, \times, \sin, \exp,$ abs is identically zero.
- Information. Decide whether a given string can be compressed further.

All these problems are computably undecidable. The proofs generally proceed by reducing the halting problem to each of them.

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John Stillwell [Sti22, p. 370]

"The problem of deciding whether a given machine halts of a given input—the so-called **halting problem**—must be unsolvable. This result was also observed by Turing (1936)."

Graham Priest [Pri17, pp. 105-107]

"Is there an algorithm we can apply to a program (or, more precisely, its code number) and inputs, to determine whether or not a computation with that program and those inputs terminates? The answer is *no*. And this is what Turing proved. (...) The result is known as the *Halting Theorem*"

Scott Aaronson [Aar13, p. 21], [Aar99]

"Turing's first result is the existence of a "universal" machine (...) But this result is not even the main result of the paper. So what is the main result? It's that there's a basic problem, called the halting problem, that no program can ever solve."

"Turing proved that this problem, called the Halting Problem, is unsolvable by Turing machines. The proof is a beautiful example of self-reference. It formalizes an old argument about why you can never have perfect introspection: because if you could, then you could determine what you were going to do ten seconds from now, and then do something else. Turing imagined that there was a special machine that could solve the Halting Problem. Then he showed how we could have this machine analyze itself, in such a way that it has to halt if it runs forever, and run forever if it halts."

Thomas Cormen [Cor13, p. 210]

"(T)here are problems for which it is provably impossible to create an algorithm that always gives a correct answer. We call such problems ***undecidable***, and the best-known one is the ***halting problem***, proven undecidable by the mathematician Alan Turing in 1937. In the halting problem, the input is a computer program A and the input *x* to A. The goal is to determine whether program A, running on input *x*, ever halts."

Dexter Kozen [Koz12, pp. 243-244]

"The technique of diagonalization was first used by Cantor to show that there are fewer real algebraic numbers than real numbers. Universal Turing Machines and the application of Cantor's diagonalization technique to prove the undecidability of the halting problem appear in Turing's original paper."

Oron Shagrir [Sha06, p. 3]

"[In his 1936 paper] Turing provides a mathematical characterization of his machines, proves that the set of these machines is enumerable, shows that there is a universal (Turing) machine, and describes it in detail. He formulates the halting problem, and proves that it cannot be decided by a Turing machine. On the basis of that proof, Turing arrives, in section 11, at his ultimate goal: proving that the *Entscheidungsproblem* is unsolvable."

Douglas Hofstadter [Hof04, p. XII]

"Fully to fathom even one other human being is far beyond our intellectual capacity — indeed, fully to fathom even one's own self is an idea that quickly leads to absurdities and paradoxes. This fact Alan Turing understood more deeply than nearly anyone ever has, for it constitutes the crux of his work on the halting problem."

Roger Penrose [Pen94, p. 30, our emphasis]

“The mathematical proofs that Hilbert’s tenth problem and the tiling problem are not soluble by computational means are difficult, and I shall certainly not attempt to give the arguments here. The central point of each argument is to show, in effect, how any Turing-machine action can be coded into a Diophantine or tiling problem. *This reduces the issue to one that Turing actually addressed in his original discussion: the computational insolubility of the halting problem*.”

Piergiorgio Odifreddi [Odi92, p. 150]

“The name [of the following theorem] comes from its original formulation, which was in terms of Turing machines, and in that setting it shows that there is no Turing machine that decides whether a universal Turing machine halts or not on given arguments.

Theorem II.2.7 Unsolvability of the Halting Problem (Turing [1936]) *The set defined by* $\langle x, e\rangle \in \mathcal{K}_0 \leftrightarrow x \in \mathcal{W}_e \leftrightarrow \varphi_e(x)\downarrow$ *is r.e. and nonrecursive.*”

Hartley Rogers, Jr

[RJ87, p. 19]. "There is no effective procedure by which we can tell whether or not a given effective computation will eventually come to a stop. (Turing refers to this as the unsolvability of the halting problem for machines. This and the existence of the universal machine are the principal results of Turing's first paper.)"

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Joel David Hamkins [Ham21, §6.5]

“Is the halting problem computably decidable? In other words, is there a computable procedure, which on input (p, n) will output yes or no depending on whether program p halts on input n? The answer is no, there is no such computable procedure; the halting problem for Turing machines is not computably decidable. This was proved by Turing with an extremely general argument, a remarkably uniform idea that applies not only to his machine concept, but which applies generally with nearly every sufficiently robust concept of computability.”

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A historical oddity

Almost none of the things attributed to Turing in those quotations are to be found in Turing's paper.

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- He does not remark on self-contemplative Turing machines;
- Nothing like the self-referential proof of undecidability is found;
- All his undecidability arguments instead use the circle-free problem, not even computably equivalent to the halting problem.

The circle-free problem

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A program is circle-free, when it succeeds in giving us infinitely many digits of a computable real number.

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Namely, if circle-free were decidable, we could computably generate a list of all computable real numbers.

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Namely, if circle-free were decidable, we could computably generate a list of all computable real numbers.

And then we could diagonalize against this list to create a real number that is not on the list.

Since our process is computable, this is a contradiction! □

Circle-free problem is strictly harder than the halting problem

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So it is a strictly weaker result to show that it is undecidable.

The printing problem

Turing also proves that the *printing problem* is undecidable.

Printing problem

Decide if a given program will ever print a certain symbol during its computation.

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Printing problem

Decide if a given program will ever print a certain symbol during its computation.

Turing gives a very clever argument for this.

- It is not a reduction of the circle-free problem to the printing problem (and indeed this is impossible)
- Rather, it is a reductio: if printing problem were decidable, then so to would be the circle-free problem.

Alternative halting criterion

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But it could easily have been formalized by printing a certain special *halt* symbol.

So the halting problem is easily seen as essentially equivalent to the printing problem.

We might all have been talking about the printing problem everywhere instead of the halting problem.

Self-referential proof of undecidability for printing

We can mimic the self-referential proof with the printing problem.

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The printing problem is computably undecidable

Assume toward contradiction that the printing problem is decidable. Consider algorithm q which on input p, a program, asks whether p on input p would ever print $\downarrow$. If so, then q halts without printing $\downarrow$; but if not, then q prints $\downarrow$ immediately. So q has opposite behavior on input p than p has on input p. So q on input q prints $\downarrow$ if and only if it does not. Contradiction.

Mathematical attribution practice

A cultural observation: mathematicians are often generous in their attributions.

They attribute results, ideas, methods to earlier thinkers, even when those thinkers didn't actually quite do it, but the ideas grow directly out of the earlier work.

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Turing didn't consider the halting problem, but he provided all the tools and ideas that we need to prove undecidability ourselves.

Attribution examples

1. Irrationality of $\sqrt{2}$ attributed to the Pythagoreans.
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5. Fundamental theorem of finite games attributed to Zermelo 1913.
6. Gödel is credited with improved strong versions of incompleteness theorem.
7. Hilbert spaces.

A nuanced conclusion

Strictly speaking, Turing did not prove nor even state the undecidability of the halting problem in his 1936 paper [Tur36].

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Nevertheless, Turing provided a robust framework of ideas sufficient to lead to the undecidability result.

And he proved the undecidability of the printing problem, easily viewed today as computably equivalent to the halting problem.

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Nevertheless, Turing provided a robust framework of ideas sufficient to lead to the undecidability result.

And he proved the undecidability of the printing problem, easily viewed today as computably equivalent to the halting problem.

In light of this, we find it correct to say:

> *Turing essentially proved the undecidability of the halting problem in [Tur36].*

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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