The elementary theory of surreal arithmetic is bi-interpretable with set theory

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The elementary theory of surreal arithmetic

This talk is based in part on very new joint work in progress by myself with Junhong Chen and Ruizhi Yang (Fudan University, Shanghai).

The surreal numbers

Let us endeavor together to build the numbers, ALL the numbers, great and small.

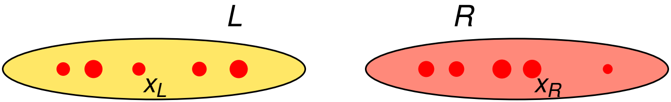
The *surreal* number system, a monumental ediface of numbers, encompasses the natural numbers, the integers, the rational numbers, the real numbers, the ordinals, the infinitesimals, and all the strange new numbers that arise in combination:

$$\sqrt{\omega} - \frac{\pi}{\omega^2}$$

The surreal numbers unify these number systems into one colossal, yet coherent and graceful number system.

Surreal number generation rule

The surreal numbers grow from nothing by the endless transfinite recursive application of a single elegant rule



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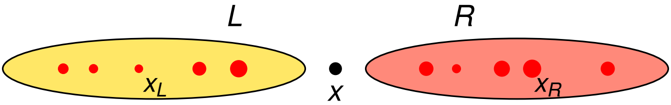

Surreal number generation rule: Iteratively fill all gaps.

At every stage, in all possible ways, separate the numbers constructed so far into a lower set and an upper set and create a new surreal number sitting strictly between them, filling this gap.

$$x = \{ L \mid R \}$$

The surreal genesis, the big bang of numbers

At the beginning, we have created no numbers yet. So we divide them into the empty lower set and empty upper set.

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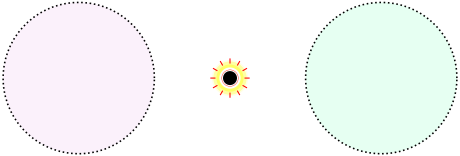

According to the surreal number generation rule, a new number is created to fill this gap.

Thus is born the first surreal number, the number *zero*, born on day 0.

$$0 = \{\ \mid\ \}.$$

Day 1

Having created 0, we now specify gaps relative to it.

$$1 = \{\, 0 \mid \ \,\}$$

also

$$-1 = \{\ \mid 0\,\}.$$

Thus altogether so far we have created three surreal numbers.

-1 **0** 1

Day 2

We proceed immediately to make four cuts in these numbers, thereby creating four new surreal numbers, born on day 2:

$$-1 \qquad \mathbf{0} \qquad 1$$

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$$-2 \quad -1 \quad -\frac{1}{2} \quad \mathbf{0} \quad \frac{1}{2} \quad 1 \quad 2$$

$$-2 = \{\ \mid -1\,\} \qquad \frac{1}{2} = \{\,0 \mid 1\,\} \qquad 2 = \{\,1 \mid\ \}$$

Day 3 . . .

On day 3 we fill the new gaps.

$$-3\ -2\ -\tfrac{3}{2}\ -1\ -\tfrac{3}{4}\ -\tfrac{1}{2}\ -\tfrac{1}{4}\ \mathbf{0}\ \tfrac{1}{4}\ \tfrac{1}{2}\ \tfrac{3}{4}\ 1\ \tfrac{3}{2}\ 2\ 3$$

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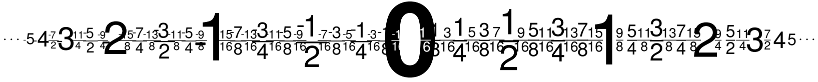
Thus, the surreal number line emerges.



On finite days exactly the dyadic rationals are born.

Day ω

Day ω is a festive day—many new numbers are born.

- all the real numbers r, filling cuts in the dyadic rationals

$$1/3 \qquad 5/7 \qquad \sqrt{2} \qquad e \qquad \pi$$

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- $\frac{1}{8} + \varepsilon = \{\, \frac{1}{8} \mid \frac{1}{8} + \frac{1}{2^n} \,\}$
- $q + \varepsilon = \{\, q \mid q + 1/2^n \,\}$ for each dyadic rational q
- $q - \varepsilon$

Further stages

Day $\omega + 1$

- $\pi + \varepsilon$

Further stages

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- $\pi + \varepsilon$
- $\sqrt{2} - \varepsilon$

Further stages

Day $\omega+1$

- $\pi+\varepsilon$
- $\sqrt{2}-\varepsilon$
- $q+\frac{\varepsilon}{2}$ for dyadic rational q

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Day $\omega + 1$

- $\pi + \varepsilon$
- $\sqrt{2} - \varepsilon$
- $q + \frac{\varepsilon}{2}$ for dyadic rational q
- $q + 2\varepsilon$
- $\pm(\omega + 1)$
- $\omega - 1 = \{\ 0 \quad 1 \quad 2 \quad \cdots \mid \omega\ \}$

At every ordinal birthday, new surreal numbers are created on top, at the bottom, and in between any conceivable cut in the numbers previously created.

Surreals are a proper class

The surreal numbers accumulate as the days pass endlessly through the transfinite hourglass of time.

$$\mathbb{N}\mathsf{o} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{N}\mathsf{o}_\alpha$$

The surreals $\mathbb{N}\mathsf{o}$ form a proper class, stratified by the sets $\mathbb{N}\mathsf{o}_\alpha$, consisting of the numbers born before day α.

Surreal numerals

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My way of constructing the surreal numbers defines the order as the numbers are created, using the gap-filling idea.

An alternative standard method has surreal numbers as special case of Conway games, defining order $x \leq y$ from numerals—it holds when there is no instance of $y \leq x_L$ or $y_R \leq x$.

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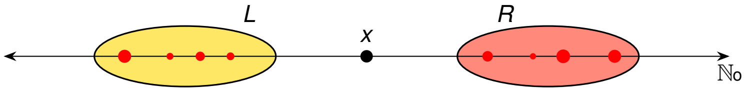

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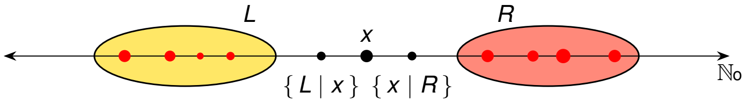

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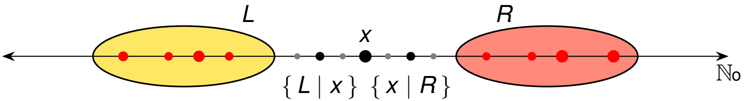

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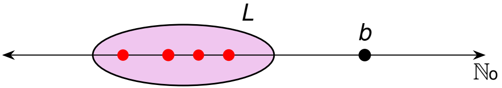

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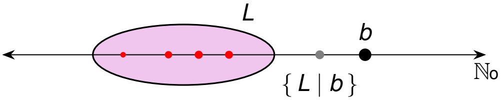

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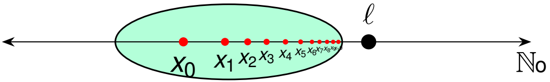

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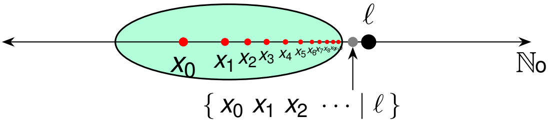

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In fact, every set of surreal numbers is discrete.

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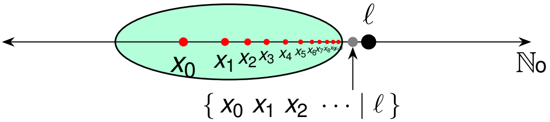

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A red flag for calculus?

Universality

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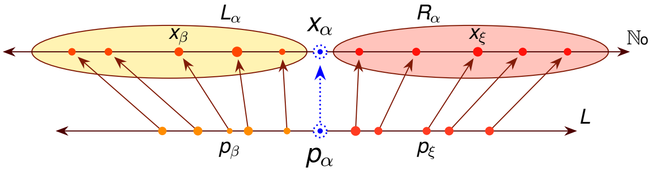
Back-and-forth argument. Enumerate your order in a well-ordered sequence, define the embedding $p_\alpha \mapsto x_\alpha$ by recursion, taking the born number filling the gap.



Surreals are also universal with respect to their algebraic structure.

The simplicity order

There is a natural tree structure underlying the surreal numbers.

Definition

Surreal x is *simpler* than y, written $x \sqsubseteq y$, if y sits in the gap defining x.

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In a sense, x is simpler than y, if you naturally construct x along the way to constructing y.

This is a tree order, arising from these nested gaps.

Surreal tree

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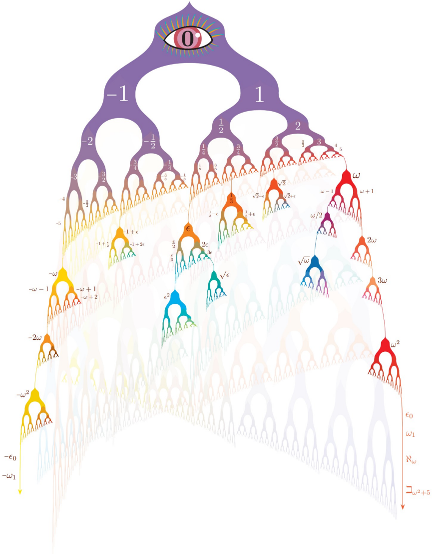

Sign sequence representation

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$$
\begin{aligned}
0 &= \langle\rangle \\
1 &= \langle+\rangle \\
-1 &= \langle-\rangle \\
2 &= \langle++\rangle \\
-2 &= \langle--\rangle \\
\tfrac{1}{2} &= \langle+-\rangle \\
\tfrac{3}{2} &= \langle++-\rangle \\
\omega &= \langle+++++\cdots\rangle \\
\alpha &= \langle+++++\cdots+\cdots\rangle \\
\varepsilon &= \langle+----\cdots\rangle
\end{aligned}
$$

Ordinals in the surreals

The ordinals arise naturally in the surreals.

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Ordinal α is the largest number born on day α.

The ordinal surreal numbers are exactly those with an empty right set $\{\, L \mid \ \}$, the first-born number bigger than a given set of surreal numbers.

Surreal addition

We define surreal addition recursively

$$x + y = \{\, x + y_L \quad x_L + y \mid x + y_R \quad x_R + y \,\}.$$

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$$x + y = \{\, x + y_L \quad x_L + y \mid x + y_R \quad x_R + y \,\}.$$

The idea is that $x + y$ should be larger than every $x_L + y$ and $x + y_L$ and smaller than every $x + y_R$ and $x_R + y$.

Specifies $x + y$ in terms of simpler instances, so this is a well founded recursion.

“Genetic definition”

Surreal multiplication

Similarly, we define surreal multiplication by

$$x \cdot y = \{\, x_L y + x y_L - x_L y_L \quad x_R y + x y_R - x_R y_R \mid x_L y + x y_R - x_L y_R \quad x y_L + x_R y - x_R y_L \,\}$$

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This is motivated by the idea that we want

$$0 < (x - x_L)(y - y_L) = xy - x_L y - x y_L + x_L y_L$$

and consequently

$$x_L y + x y_L - x_L y_L < xy.$$

The surreal field

One can now prove that the surreal numbers form an ordered field $\langle \text{No}, +, \cdot, 0, 1, < \rangle$, indeed, a real-closed field.

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One can now prove that the surreal numbers form an ordered field $\langle \mathbb{N}\mathrm{o}, +, \cdot, 0, 1, < \rangle$, indeed, a real-closed field.

The surreal field has the capacity to serve as the hyperreals in nonstandard analysis.

Mathematicians seek to find natural genetic definitions for other analytic functions and thereby to found calculus and analysis on the surreal numbers.

Several fun challenge problems

I should like to mention a series of fun challenge problems that explore set-theoretic and topological aspects of the surreal numbers.

How many unit intervals in $[0, \omega]$

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But...

A proper class of disjoint unit intervals fit in $[0, \omega]$

Descend from the top, forming $\omega - 1$, $\omega - 2$, and so forth. At limits, there is a number in the gap, so you can keep going for Ord many steps. In fact, can get $\mathbb{N}\mathrm{o}$ many, since you can keep fitting copies of $\mathbb{Z}$ in between other copies, filling gaps just as in $\mathbb{N}\mathrm{o}$.

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Yes, we can move π, and in fact, the orbit takes it to any desired element of the infinitesimality class of π. Reason: $\mathbb{N}\mathrm{o}$ is a saturated real-closed field, and all such numbers realize the same 1-type. So we can extend via back-and-forth.

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Similarly, we can move ω to $\sqrt{\omega} + 17$ or to $\omega_5 - \frac{1}{2}\omega^2$.

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Global choice.

Is birthday notation $\{ L \mid R \}$ structural?

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Answer

No. The automorphisms moving ω fix every natural number, but not $\{\, 0, 1, 2, \ldots \mid \quad \}$.

The birthday structure is not part of the field structure, but strictly expands it.

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As a field, the theory is decidable, but with the birthday structure, we can interpret arithmetic.

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Answer 1. No

Cover the unit interval with intervals of size ε. No finitely many of them can cover. No continuum-sized subcover, since each contains at most one real. Actually, no set-sized subcover. Not set-Lindelöf.

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Answer 1. No (class covers)

Cover the unit interval with intervals of size ε. No finitely many of them can cover. No continuum-sized subcover, since each contains at most one real. Actually, no set-sized subcover. Not set-Lindelöf.

Answer 2. Yes (set covers)

Meanwhile, every cover of the unit interval with a *set* of open intervals admits a finite subcover. Proof: Cover $[0,1]$ with a set of intervals. Let L have all x appearing as an endpoint such that $[0,x]$ is covered by a finite subcover. Let R be the left end-points of the remaining intervals. Now $\{ L \mid R \}$ is not covered. □

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Answer 1

The surreals are totally disconnected with respect to class unions of intervals. Just cut the line at an $(\mathrm{Ord}, \mathrm{Ord})$ gap between the two numbers.

Answer 2

Meanwhile, there is no disconnection using *sets* of open intervals. Use the same idea as in compactness.

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Answer 1

Let *f* be the first-born number in each infinitesimality class. This is locally constant, hence continuous according to the $\varepsilon\delta$ definition of continuity. But it fails the IVT property.

Intermediate value theorem

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Does the IVT hold for the surreals?

Answer 1

Let f be the first-born number in each infinitesimality class. This is locally constant, hence continuous according to the $\varepsilon\delta$ definition of continuity. But it fails the IVT property.

Answer 2

Suppose f is continuous in that f^{-1} of an open interval is a *set* union of open intervals. Then actually IVT holds for f. For otherwise $f^{-1}(-\infty, d) \cup f^{-1}(d, \infty)$ would be a disconnection of $\mathbb{N}\mathrm{o}$.

Global choice

Several features of the surreal numbers rely on global choice

- Universality of the order, field structure
- Automorphisms via back-and-forth

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Global choice

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- Universality of the order, field structure
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Thus, the theorems are often stated in Gödel-Bernays set theory GBC, which has global choice.

Years ago, I asked whether one can prove the universality of the surreals in ZFC for definable linear orders? I suspect not, but this remains open.

Surreal structure

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- Surreal order $<$
- birthday order $\prec$
- same-birthday relation $\approx$
- left-prior relation $a \mathrel{\text{L}} x$
- right-prior relation $x \mathrel{\text{R}} b$
- canonical numerals $x = \{ L_x \mid R_x \}$
- field structure $x + y$, $x \cdot y$, 0, 1
- simplicity order $\sqsubseteq$
- sign sequence $x(\alpha) = +/-$

Surreal field

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From this point of view, we do not conceive of the surreal numbers merely as an ordered field.

Order structures

Several of the order structures are definitionally equivalent:

- $\langle \mathbb{N}\mathsf{o}, <, \prec \rangle$
- $\langle \mathbb{N}\mathsf{o}, \mathsf{L}, \mathsf{R} \rangle$
- $\langle \mathbb{N}\mathsf{o}, <, \mathsf{L} \rangle$
- $\langle \mathbb{N}\mathsf{o}, <, \mathsf{R} \rangle$
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Warning: absent from this result is $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$

Simplicity structure

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Philip Ehrlich has emphasized the core, central importance of the simplicity tree order and the structure $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$.

Indeed, using just the order $<$ and simplicity $\sqsubseteq$, we can define all the other structure in second-order logic

- We can mount all the genetic definitions
- We can define the birthday structure
- We can define $+$ and $\cdot$ and all the analytic definitions

In this respect, $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$ is a canonical fundamental surreal structure.

Weakness of simplicity in first-order logic

Nevertheless, in first-order logic $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$ is weak.

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Proof.

Produce a sufficiently saturated model of this theory with a nontrivial automorphism moving ordinal levels. So it changes birthdays. Apply the automorphism to the model, but only on the positive numbers. This preserves $<$ and $\sqsubseteq$, but not the birthday order nor $+$ or $\cdot$, so these cannot be definable. □

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I believe that a QE result is possible for $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$ in such a way that reveals the theory is trivial.

Analogue with Dedekind arithmetic

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Namely, Dedekind showed that all the usual arithmetic structure is definable in $\langle \mathbb{N}, 0, S \rangle$ using second-order logic.

But this structure is weak in first-order logic—it has a trivial, decidable theory.

Similarly, in the surreals, $\langle \mathbb{N}\mathrm{o}, <, \sqsubseteq \rangle$ is fully powerful in second-order logic, but trivial as a first-order structure.

Standard structure of surreal arithmetic

In light of those considerations, we take the standard structure of surreal arithmetic to be

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$$\langle \mathbb{N}\mathrm{o}, +, \cdot, <, \prec \rangle.$$

Augment the ordered field structure with the birthday structure.

Definitionally equivalent to many other variations, such as

$$\langle \mathbb{N}\mathrm{o}, +, \cdot, <, \mathsf{L}, \mathsf{R} \rangle.$$

We adopt all the further structure via definitional expansions.

Defined notions in surreal arithmetic

We build up defined concepts in $\langle \mathbb{N}\mathrm{o}, +, \cdot, <, \prec \rangle$

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Defined notions in surreal arithmetic

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- Define all the order and birthday structure
- α is an ordinal iff largest with that birthday
- define usual ordinal arithmetic from surreal arithmetic (these are different) via coding with Cantor normal form
- define $x \restriction \alpha$ = level α simpler approximation
- define $x(\alpha) = +/-$ depending on $x \restriction \alpha < x$ or $x < x \restriction \alpha$
- pairing function $\langle x, y \rangle$ concatenate sequences

Surreal structure bi-interpretable with V

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Theorem

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Surreal structure bi-interpretable with V

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Theorem

The surreal structure $\langle \text{No}, +, \cdot, <, \prec \rangle$ *is bi-interpretable with the full set-theoretic universe* $\langle V, \in \rangle$*.*

- mutually interpretable—each structure defines a copy of the other (modulo a definable equivalence relation)
- bi-interpretable—each can also define the isomorphism of itself with the copy of itself in the interpreted structure

Mutual interpretation

Interpreting № in V

This is the easy direction, since the surreal structure is defined in set theory.

Mutual interpretation

Interpreting $\mathbb{N}o$ in V

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Interpreting V in $\mathbb{N}o$

Every set x is an element of some transitive set X, and so there is cardinal κ with extensional well-founded $E \subseteq \kappa \times \kappa$ such that $\langle X, \in, x\rangle \cong \langle \kappa, E, \alpha\rangle$. So x is represented by E and α, naturally coded as a set of ordinals.

But every set of ordinals is coded by a surreal number via the sign sequences. In the surreals, we can do this coding. We can express when two codes are the same, and when one code represents an element of another.

Bi-interpretation

For the bi-interpretation claim, we observe that V can see fully how each of its sets x are represented inside $\mathbb{No}$ by a surreal number coding a binary sequence that gives a pointed extensional well-founded relation on an ordinal.

Conversely, the surreal structure can see how sets are interpreted, and inside the resulting model of set theory there is the class of surreal numbers, and in the parent surreals we can associate every surreal number r with the surreal number $\hat{r}$ that represents the very same surreal number as a set within the encoding of V by surreals. The two surreals will have the same sign sequence.

A first-order theory of surreal arithmetic

Finally, we should like to introduce the first-order elementary theory of surreal arithmetic SA, stated in the language of

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The final goal is to prove that SA is bi-interpretable with ZFC. Let me tell you the axioms.

Extensionality

Axiom of extensionality

Any two surreal numbers with the same left-prior and right-prior sets are equal.

Surreal arithmetic operations

Addition recursion

We express that $+$ obeys the recursive definition

$$x + y = \{\, x + y_L \quad x_L + y \mid x + y_R \quad x_R + y \,\}.$$

This is expressible in the language of surreal arithmetic.

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This is expressible in the language of surreal arithmetic.

Multiplication recursion

We similarly express that multiplication obeys its recursion

$$x \cdot y = \{\, x_L y + x y_L - x_L y_L \quad x_R y + x y_R - x_R y_R \mid x_L y + x y_R - x_L y_R \quad x y_L + x_R y - x_R y_L \,\}$$

Surreal arithmetic operations

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This is like the PA axioms stipulating the recursive definitions for addition and multiplication.

Order axioms

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Axiom of foundation

The birthday order $\prec$ is definably well-founded. Every nonempty definable class has a first-born member.

Definable saturation

Axiom of definable saturation (scheme)

For any formulas φ_L and φ_R, if

$$\varphi_L(a) \wedge \varphi_R(b) \implies a < b$$

and furthermore all such instances a, b have their birthdays uniformly bounded, then there is x with that birthday such that $a < x < b$ for all such instances.

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and furthermore all such instances a, b have their birthdays uniformly bounded, then there is x with that birthday such that $a < x < b$ for all such instances.

This axiom expresses that every definable gap is filled, a first-order definable version of the core underlying surreal idea.

Eternity

Axiom of eternity (scheme)

The birthdays proceed longer than we can describe. Namely, for any ordinal γ and any definable map $\alpha \mapsto r_\alpha$, where $\alpha < \gamma$, there is a birthday beyond all r_α.

Exponential power

Exponential power axiom

For any ordinal α, there is a surreal number *r* whose sign sequence consists of all possible sign-sequences of length α concatenated together.

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This will help give us the power set axiom.

Bi-interpretation

Main ideas

- Equivalence of codes gives extensionality of the coded sets.
- Saturation gives pairing, union, separation. We can define the desired codes, so they exist in the interpreted structure.
- Eternity implies the collection axiom for ordinals in the interpreted set universe. Hence replacement.
- Can prove that $+, \cdot$ form a real-closed field. Use the coding to show that genetically-defined functions have solutions.
- Axiom of choice is free in the interpreted set universe, since every coded set in $\mathbb{N}\mathrm{o}$ in effect comes along with a coded well-order of it.

Very new work, details still being checked.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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