

Epistemic Logic Puzzles for Mathematicians

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

NYC Mathematics Education and Communication Seminar
October 20, 2026

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

First logician: *I don't know.*

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

First logician: *I don't know.*

Second logician: *I don't know.*

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

First logician: *I don't know.*

Second logician: *I don't know.*

Third logician: *Yes.*

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

First logician: *I don't know.*

Second logician: *I don't know.*

Third logician: *Yes.*

Question

How many beers does the waiter bring?

The Blue-Eyed Islanders

A remote island has 100 perfectly logical inhabitants, each with vivid deep blue eyes.

They never discuss eye color. Further, if any of them should learn their own eye color, they leave the island next dawn in a flashy display.



The Blue-Eyed Islanders

A remote island has 100 perfectly logical inhabitants, each with vivid deep blue eyes.

They never discuss eye color. Further, if any of them should learn their own eye color, they leave the island next dawn in a flashy display.



A trusted friend visits the island. At her departure, in front of the entire community, she remarks:

At least one of you has blue eyes.

The Blue-Eyed Islanders

A remote island has 100 perfectly logical inhabitants, each with vivid deep blue eyes.

They never discuss eye color. Further, if any of them should learn their own eye color, they leave the island next dawn in a flashy display.



A trusted friend visits the island. At her departure, in front of the entire community, she remarks:

At least one of you has blue eyes.

Exactly 100 days later, all the islanders depart at once in a flashy display.

Why?

Blue-eyed islanders solution

Work up from easier instances.

Blue-eyed islanders solution

Work up from easier instances.

One islander. If there is only one blue-eyed islander, he learns his eye color immediately and departs next dawn.

Blue-eyed islanders solution

Work up from easier instances.

One islander. If there is only one blue-eyed islander, he learns his eye color immediately and departs next dawn.

Two islanders. Each knows the other is blue; but unsure of their own color. They reason: If I'm not blue, then the other islander will leave tomorrow. But this doesn't occur—each is waiting for the other. So both know then, depart on 2nd day.

Blue-eyed islanders solution

Work up from easier instances.

One islander. If there is only one blue-eyed islander, he learns his eye color immediately and departs next dawn.

Two islanders. Each knows the other is blue; but unsure of their own color. They reason: If I'm not blue, then the other islander will leave tomorrow. But this doesn't occur—each is waiting for the other. So both know then, depart on 2nd day.

Three islanders. Each sees 2 blue-eyed islanders. If those were all, they would depart 2nd day. This doesn't occur, so all three realize they are blue and depart on day 3.

Blue-eyed islanders solution

Work up from easier instances.

One islander. If there is only one blue-eyed islander, he learns his eye color immediately and departs next dawn.

Two islanders. Each knows the other is blue; but unsure of their own color. They reason: If I'm not blue, then the other islander will leave tomorrow. But this doesn't occur—each is waiting for the other. So both know then, depart on 2nd day.

Three islanders. Each sees 2 blue-eyed islanders. If those were all, they would depart 2nd day. This doesn't occur, so all three realize they are blue and depart on day 3.

By induction, n blue-eyed islanders depart at dawn of day n .

All 100 islanders depart at dawn 100th day in a flashy display.

A subtle philosophical conundrum

Observation

When the visitor says

“At least one of you has blue eyes,”

this is something that each islander knows already.

They didn't seem to learn any new information.

A subtle philosophical conundrum

Observation

When the visitor says

“At least one of you has blue eyes,”

this is something that each islander knows already.

They didn't seem to learn any new information.

Question

How is it possible that being told something they already knew enabled them to deduce their own eye color?

The Pirate-Treasure Division Problem

Pirate ship with a fearsome, perfectly logical crew. The pirate treasure of 100 gold coins is to be divided.



Pirate-treasure division protocol. Pirates are ranked—captain, first lieutenant, second lieutenant. Let's call them pirate 1, pirate 2, etc.

- Lowliest pirate mounts plank, makes proposal.
- All pirates vote.
- If majority approve, that's the plan.
- If not, the pirate walks the plank. Next lowliest pirate mounts plank.

The Pirate-Treasure Division Problem

Pirate ship with a fearsome, perfectly logical crew. The pirate treasure of 100 gold coins is to be divided.



Pirate-treasure division protocol. Pirates are ranked—captain, first lieutenant, second lieutenant. Let's call them pirate 1, pirate 2, etc.

- Lowliest pirate mounts plank, makes proposal.
- All pirates vote.
- If majority approve, that's the plan.
- If not, the pirate walks the plank. Next lowliest pirate mounts plank.

Question

You are pirate 10. What is your proposal?

Pirate value system

Strictly ordered with priority.

- A Most important, stay alive.
- B Get gold—the more the better.
- C Cause the death of other pirates, if possible.
- D All other things being equal, arrange when possible that gold goes to senior pirates.

Pirate treasure division

Question

What is your plan as pirate 10?

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes. No matter the proposal, captain will vote against, in light of pirate value C. Regrettably, pirate 2 will die.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes. No matter the proposal, captain will vote against, in light of pirate value C. Regrettably, pirate 2 will die.

Three pirates.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes. No matter the proposal, captain will vote against, in light of pirate value C. Regrettably, pirate 2 will die.

Three pirates. Pirate 3 needs 2 votes.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes. No matter the proposal, captain will vote against, in light of pirate value C. Regrettably, pirate 2 will die.

Three pirates. Pirate 3 needs 2 votes. Pirate 2 will vote in favor, since otherwise dead.

Pirate treasure division

Question

What is your plan as pirate 10?

Naturally, pirates will compare with expected pirate 9 plan.

So we should work backwards.

One pirate. If there is only the captain, he mounts plank, proposes “I get all the gold,” votes in favor, and it succeeds.

Two pirates. Pirate 2 mounts plank. Needs majority of 2—both votes. No matter the proposal, captain will vote against, in light of pirate value C. Regrettably, pirate 2 will die.

Three pirates. Pirate 3 needs 2 votes. Pirate 2 will vote in favor, since otherwise dead. So pirate 3 proposes, “I get all the gold.” Approved by 2,3.

Four pirates.

Four pirates. Pirate 4 needs three votes.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Five pirates.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Five pirates. Needs three votes.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Five pirates. Needs three votes. Can buy pirate 3's vote for 1 gold.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Five pirates. Needs three votes. Can buy pirate 3's vote for 1 gold. Can buy pirate 1's or 2's vote with 2 gold.

Four pirates. Pirate 4 needs three votes. Pirates 1 and 2 get nothing in pirate 3 plan. So propose: “1 gold each to pirates 1 and 2, and 98 gold for myself.” Passes with approval of pirates 1, 2, 4.

Five pirates. Needs three votes. Can buy pirate 3's vote for 1 gold. Can buy pirate 1's or 2's vote with 2 gold. In light of pirate value D, prefer pirate 1. So: 2 gold to 1; 1 gold to 3; and 97 gold to me. Passes.

	1	2	3	4	5	6	7	8	9	10
One pirate	100									
Two pirates	*	X								
Three pirates	0	0	100							
Four pirates	1	1	0	98						
Five pirates	2	0	1	0	97					
Six pirates	0	1	2	1	0	96				
Seven pirates	1	2	0	0	1	0	96			
Eight pirates	2	0	1	1	0	1	0	95		
Nine pirates	0	1	2	0	1	0	1	0	95	
Ten pirates	1	2	0	1	0	1	0	1	0	94

The Philosopher's Ruling Council

In land of Philosophia, there is a ruling council of philosophers.

Philosopher 1 is the philosopher king, followed by philosopher 2, philosopher 3, and so on.

It is time to appoint a new ruling council, all perfectly logical.

Procedure:

The Philosopher's Ruling Council

In land of Philosophia, there is a ruling council of philosophers.

Philosopher 1 is the philosopher king, followed by philosopher 2, philosopher 3, and so on.

It is time to appoint a new ruling council, all perfectly logical.

Procedure:

- Lowest rank member proposes a new council+ranking.
- New members can be anyone from Philosophia.
- Current council votes.
- If majority approve, that is the new ruling council.
- If not, member is ejected from council. New lowest member makes proposal.

Philosopher values

This is a power-hungry, Machiavellian crowd.

Every philosopher:

- Prefers being on the new council to all other things.
- Will never vote to approve a council they are not on.
- Prefers being as high on new council as possible.
- Given that, prefers council to be as small as possible.

Philosopher values

This is a power-hungry, Machiavellian crowd.

Every philosopher:

- Prefers being on the new council to all other things.
- Will never vote to approve a council they are not on.
- Prefers being as high on new council as possible.
- Given that, prefers council to be as small as possible.

Question

You are philosopher 5—what is your proposal?

Philosopher's ruling council solution

Work backwards.

Philosopher's ruling council solution

Work backwards.

One philosopher.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.”

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes. Can get 1's vote for free. Can get 2's vote by making king. Proposal: 2, 4, 1. Passes.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes. Can get 1's vote for free. Can get 2's vote by making king. Proposal: 2, 4, 1. Passes.

Five philosophers.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes. Can get 1's vote for free. Can get 2's vote by making king. Proposal: 2, 4, 1. Passes.

Five philosophers. Need 3 votes.

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes. Can get 1's vote for free. Can get 2's vote by making king. Proposal: 2, 4, 1. Passes.

Five philosophers. Need 3 votes. Proposal:

Philosopher's ruling council solution

Work backwards.

One philosopher. If there is only the philosopher king, he proposes “I remain king,” and this is approved.

Two philosophers. Needs majority of 2. No proposal will be approved by philosopher 1. Rejected.

Three philosophers. Need 2 votes. Proposal: “I become king, old philosopher 2 remains philosopher 2.” Passes with approval of philosophers 2, 3.

Four philosophers. Need 3 votes. Can get 1's vote for free. Can get 2's vote by making king. Proposal: 2, 4, 1. Passes.

Five philosophers. Need 3 votes. Proposal: 5, 1, 3. Passes.

Philosopher's ruling council—winning proposals

Size of Council	Proposed New Council and Ranking
1	1
2	Proposal rejected
3	3 2
4	2 4 1
5	5 1 3

The Philosophical society

Annual entrance examination for 100 ambitious philosopher candidates to the Philosophical Society.

We all stand in a line facing the society hall.

A hat, either black or white, is placed on each head. We can see only those ahead, but not those behind or our own.

The Philosophical society

Annual entrance examination for 100 ambitious philosopher candidates to the Philosophical Society.

We all stand in a line facing the society hall.

A hat, either black or white, is placed on each head. We can see only those ahead, but not those behind or our own.

Starting at the back, each candidate will shout a guess of their hat color in turn.

If we are all correct, then we are admitted as a group, but otherwise all are sent down.

The Philosophical society

Annual entrance examination for 100 ambitious philosopher candidates to the Philosophical Society.

We all stand in a line facing the society hall.

A hat, either black or white, is placed on each head. We can see only those ahead, but not those behind or our own.

Starting at the back, each candidate will shout a guess of their hat color in turn.

If we are all correct, then we are admitted as a group, but otherwise all are sent down.

A concession: before it starts, the candidate at very back may inspect her hat and even change the color if desired.

The Philosophical society

Annual entrance examination for 100 ambitious philosopher candidates to the Philosophical Society.

We all stand in a line facing the society hall.

A hat, either black or white, is placed on each head. We can see only those ahead, but not those behind or our own.

Starting at the back, each candidate will shout a guess of their hat color in turn.

If we are all correct, then we are admitted as a group, but otherwise all are sent down.

A concession: before it starts, the candidate at very back may inspect her hat and even change the color if desired.

How well can we do?

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.
Often does well, since most people are in majority.

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.
Often does well, since most people are in majority.
But in worst case, everyone is wrong: W B W B W B . . . W B.

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.
Often does well, since most people are in majority.
But in worst case, everyone is wrong: W B W B W B . . . W B.

Choose randomly

Everyone flip a coin for the guess. This will get half right on average, but likelihood of everyone right is vanishingly small.

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.
Often does well, since most people are in majority.
But in worst case, everyone is wrong: W B W B W B . . . W B.

Choose randomly

Everyone flip a coin for the guess. This will get half right on average, but likelihood of everyone right is vanishingly small.

Guaranteed 51 or more

Odd-position people say the color of the person just in front of them; that person then says same color. This is at least half right. Plus very first person. So at least 51 correct.

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.
Often does well, since most people are in majority.
But in worst case, everyone is wrong: W B W B W B . . . W B.

Choose randomly

Everyone flip a coin for the guess. This will get half right on average, but likelihood of everyone right is vanishingly small.

Guaranteed 51 or more

Odd-position people say the color of the person just in front of them; that person then says same color. This is at least half right. Plus very first person. So at least 51 correct.

Can we do better?

A perfect solution is possible!

A perfect solution is possible!

The philosopher at the very back should look ahead and count the number of white hats ahead.

A perfect solution is possible!

The philosopher at the very back should look ahead and count the number of white hats ahead.

If this is even, then adopt a white hat, otherwise black.
Announce accordingly.

A perfect solution is possible!

The philosopher at the very back should look ahead and count the number of white hats ahead.

If this is even, then adopt a white hat, otherwise black.
Announce accordingly.

The next person knows almost all the same information, except their own hat color. So they can calculate white or black based on the announcement.

A perfect solution is possible!

The philosopher at the very back should look ahead and count the number of white hats ahead.

If this is even, then adopt a white hat, otherwise black.
Announce accordingly.

The next person knows almost all the same information, except their own hat color. So they can calculate white or black based on the announcement.

Each person pays attention to the prior announcements, and looks ahead. Knowing the parity, they can deduce their own hat color.

A perfect solution is possible!

The philosopher at the very back should look ahead and count the number of white hats ahead.

If this is even, then adopt a white hat, otherwise black.
Announce accordingly.

The next person knows almost all the same information, except their own hat color. So they can calculate white or black based on the announcement.

Each person pays attention to the prior announcements, and looks ahead. Knowing the parity, they can deduce their own hat color.

Everyone will be right!

The Physics Society

The physicists use 10 hat colors, instead of just two.

How well can they do? The even/odd parity idea doesn't seem to help much.

The Physics Society

The physicists use 10 hat colors, instead of just two.

How well can they do? The even/odd parity idea doesn't seem to help much.

Simply use base 10 instead of base 2. Each hat is a number 0 to 9, and first candidate picks the total sum mod 10.

Everyone can determine their own hat color based on what came before and the total residue.

So everyone will be correct!

The mathematical society

The mathematicians use natural numbers, written on slates.

How well can they do?

The mathematical society

The mathematicians use natural numbers, written on slates.

How well can they do?

Easy solution: first person simply announces the total sum.

Everyone can calculate their number based on the prior numbers, the observed numbers, and the total sum.

The mathematical society

The mathematicians use natural numbers, written on slates.

How well can they do?

Easy solution: first person simply announces the total sum.

Everyone can calculate their number based on the prior numbers, the observed numbers, and the total sum.

What about real numbers?

The mathematical society

The mathematicians use natural numbers, written on slates.

How well can they do?

Easy solution: first person simply announces the total sum.

Everyone can calculate their number based on the prior numbers, the observed numbers, and the total sum.

What about real numbers?

Same method works in the real numbers. Indeed, it works in any group.

Since any nonempty set can carry a group structure, the method works with any set of labels!

The Midnight Ball

A festive ball, with infinitely many guests.

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

At midnight, one of us will be selected as Queen of the Ball, allowed to erase her number and have one of her choosing.

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

At midnight, one of us will be selected as Queen of the Ball, allowed to erase her number and have one of her choosing.

Then, we all shout our best predictions for our own numbers.

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

At midnight, one of us will be selected as Queen of the Ball, allowed to erase her number and have one of her choosing.

Then, we all shout our best predictions for our own numbers.

How well can we do?

The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

At midnight, one of us will be selected as Queen of the Ball, allowed to erase her number and have one of her choosing.

Then, we all shout our best predictions for our own numbers.

How well can we do?

We are allowed to plan in advance, but no communication is allowed once the numbers are assigned.

First steps

Of course, the queen can guess correctly.

First steps

Of course, the queen can guess correctly.

Better, she can communicate a finite amount of information with her choice of number.

First steps

Of course, the queen can guess correctly.

Better, she can communicate a finite amount of information with her choice of number.

That is, we can agree on a finite set of us, and let the queen pick the number with prime factorization

$$p_0^{k_0} p_1^{k_1} \cdots p_n^{k_n},$$

which tells person i that they have k_i .

First steps

Of course, the queen can guess correctly.

Better, she can communicate a finite amount of information with her choice of number.

That is, we can agree on a finite set of us, and let the queen pick the number with prime factorization

$$p_0^{k_0} p_1^{k_1} \cdots p_n^{k_n},$$

which tells person i that they have k_i .

Thus, we can ensure that any given finite set of us is correct.

First steps

Of course, the queen can guess correctly.

Better, she can communicate a finite amount of information with her choice of number.

That is, we can agree on a finite set of us, and let the queen pick the number with prime factorization

$$p_0^{k_0} p_1^{k_1} \cdots p_n^{k_n},$$

which tells person i that they have k_i .

Thus, we can ensure that any given finite set of us is correct.

Can we do better?

A better solution

Yes!

A better solution

Yes!

We can do much better than finitely correct.

A better solution

Yes!

We can do much better than finitely correct.

In fact, with the right strategy we can ensure that we ALL guess correctly! Without exception.

Midnight ball solution

We will use the axiom of choice.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

We can each see who is correct and who differs from representative (except ourselves).

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

We can each see who is correct and who differs from representative (except ourselves).

Knowing total difference, we can calculate our number.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

We can each see who is correct and who differs from representative (except ourselves).

Knowing total difference, we can calculate our number.

We will all be correct!

Generalizations

The solution works even if we have uncountably many guests—any cardinality is fine.

Generalizations

The solution works even if we have uncountably many guests—any cardinality is fine.

Solution works with real numbers, instead of natural numbers.

Generalizations

The solution works even if we have uncountably many guests—any cardinality is fine.

Solution works with real numbers, instead of natural numbers.

Solutions works with labels from any group, of any size.

Therefore, we can succeed with labels coming from any nonempty set.

The Grand Guessing-Box Hall

One hundred fellow mathematician stand outside the Grand Guessing-Box Hall.

The Grand Guessing-Box Hall

One hundred fellow mathematician stand outside the Grand Guessing-Box Hall.

Inside, there are infinitely many wooden boxes, each containing a real number.

The Grand Guessing-Box Hall

One hundred fellow mathematician stand outside the Grand Guessing-Box Hall.

Inside, there are infinitely many wooden boxes, each containing a real number.

We enter alone in turn. In the hall, we can open boxes as we like and learn the numbers, but we must leave at least one box unopened, whose number we shall guess.

The Grand Guessing-Box Hall

One hundred fellow mathematician stand outside the Grand Guessing-Box Hall.

Inside, there are infinitely many wooden boxes, each containing a real number.

We enter alone in turn. In the hall, we can open boxes as we like and learn the numbers, but we must leave at least one box unopened, whose number we shall guess.

After the guess, the room is reset exactly the same, and the next mathematician enters.

The Grand Guessing-Box Hall

One hundred fellow mathematician stand outside the Grand Guessing-Box Hall.

Inside, there are infinitely many wooden boxes, each containing a real number.

We enter alone in turn. In the hall, we can open boxes as we like and learn the numbers, but we must leave at least one box unopened, whose number we shall guess.

After the guess, the room is reset exactly the same, and the next mathematician enters.

How well can we do? Can we ensure that any of us is correct?

Grand Guessing-Box Hall

We can plan in advance, but no communication is allowed once we start.

It may seem difficult how planning would help.

How could we possibly ensure someone is correct?

Grand Guessing-Box Hall

We can plan in advance, but no communication is allowed once we start.

It may seem difficult how planning would help.

How could we possibly ensure someone is correct?

Of course we cannot guarantee that any particular person is correct, since everything would be the same if we change the number in that box.

Grand Guessing-Box Hall

We can plan in advance, but no communication is allowed once we start.

It may seem difficult how planning would help.

How could we possibly ensure someone is correct?

Of course we cannot guarantee that any particular person is correct, since everything would be the same if we change the number in that box.

But of course, that change could affect other people's guesses.

Perhaps it still seems impossible to ensure even a single right answer.

Guessing-Box Solution

Nevertheless, we can win!

Guessing-Box Solution

Nevertheless, we can win!

Incredible as it may seem, we can ensure that 99 of us guess correctly!

Yes, 99 out of 100 will guess correctly.

Guessing-Box Solution

Nevertheless, we can win!

Incredible as it may seem, we can ensure that 99 of us guess correctly!

Yes, 99 out of 100 will guess correctly.

Our winning strategy will rely on the axiom of choice.

Agreement points

Familiar with the layout of boxes in the hall, we shall conceive of them as 100 infinite sequences of boxes.

Agreement points

Familiar with the layout of boxes in the hall, we shall conceive of them as 100 infinite sequences of boxes.

Define the notion of *eventual agreement* for any two possible sequence patterns.

Agreement points

Familiar with the layout of boxes in the hall, we shall conceive of them as 100 infinite sequences of boxes.

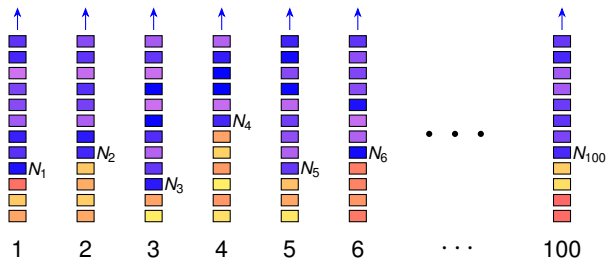
Define the notion of *eventual agreement* for any two possible sequence patterns.

By the axiom of choice, let us agree on a choice of representative for each equivalence class.

Every actual sequence will have an *agreement point*, the place where it comes into agreement with the representative.

Winning procedure

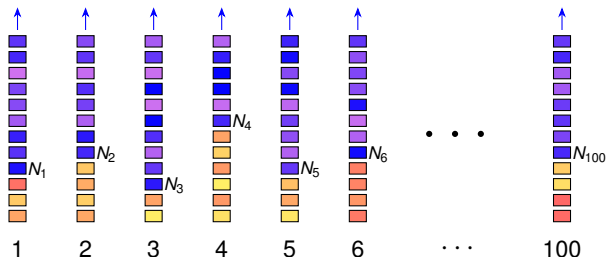
Upon entering, mathematician i opens everyone else's boxes.



She thereby learns all the other agreement points N_j for $j \neq i$.

Winning procedure

Upon entering, mathematician i opens everyone else's boxes.



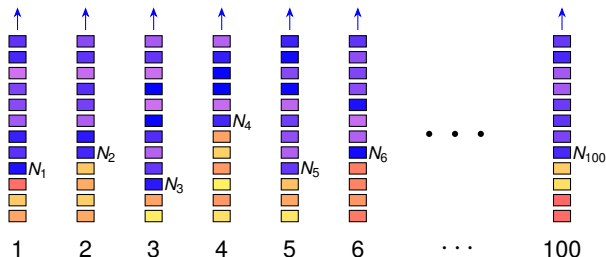
She thereby learns all the other agreement points N_j for $j \neq i$.

Takes $N = \max_{j \neq i} N_j$ as proxy for her own agreement point N_i .

She opens up her own boxes beyond N , and thereby learns her equivalence class.

Winning procedure

Upon entering, mathematician i opens everyone else's boxes.



She thereby learns all the other agreement points N_j for $j \neq i$.

Takes $N = \max_{j \neq i} N_j$ as proxy for her own agreement point N_i .

She opens up her own boxes beyond N , and thereby learns her equivalence class.

Now she guesses her box N according to the representative.

How it wins

Mathematician i will guess correctly, provided that her actual agreement point N_i is at most the proxy $\max_{j \neq i} N_j$.

How it wins

Mathematician i will guess correctly, provided that her actual agreement point N_i is at most the proxy $\max_{j \neq i} N_j$.

But only one of the mathematicians can have an agreement point that is larger than all the others.

How it wins

Mathematician i will guess correctly, provided that her actual agreement point N_i is at most the proxy $\max_{j \neq i} N_j$.

But only one of the mathematicians can have an agreement point that is larger than all the others.

Therefore, at least 99 of them will guess correctly!

At most one mathematician will be wrong.

How it wins

Mathematician i will guess correctly, provided that her actual agreement point N_i is at most the proxy $\max_{j \neq i} N_j$.

But only one of the mathematicians can have an agreement point that is larger than all the others.

Therefore, at least 99 of them will guess correctly!

At most one mathematician will be wrong.

Works just as well with 1000 mathematicians—ensure at least 999 correct.

Infinitely many guesses

At the Grander Guessing-Box Hall, each mathematician will leave infinitely many boxes unopened, making a guess about each of them.

Infinitely many guesses

At the Grander Guessing-Box Hall, each mathematician will leave infinitely many boxes unopened, making a guess about each of them.

How well can they do?

Infinitely many guesses

At the Grander Guessing-Box Hall, each mathematician will leave infinitely many boxes unopened, making a guess about each of them.

How well can they do?

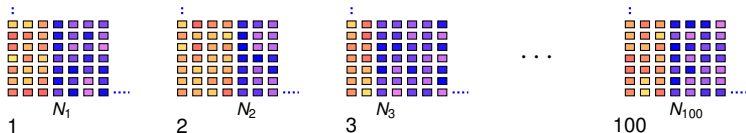
Nevertheless, we can ensure that at least 99 of them are totally correct in all their guesses.

Infinitely many guesses

At the Grander Guessing-Box Hall, each mathematician will leave infinitely many boxes unopened, making a guess about each of them.

How well can they do?

Nevertheless, we can ensure that at least 99 of them are totally correct in all their guesses.



Each mathematician gets a matrix. Use eventual agreement in the columns.

Infinitely Many Mathematicians

What if we have infinitely many mathematicians?

Infinitely Many Mathematicians

What if we have infinitely many mathematicians?

Amazing fact

For any infinite cardinal κ , suppose

- κ many mathematicians
- κ many boxes
- Each mathematician guesses about κ many boxes

Infinitely Many Mathematicians

What if we have infinitely many mathematicians?

Amazing fact

For any infinite cardinal κ , suppose

- κ many mathematicians
- κ many boxes
- Each mathematician guesses about κ many boxes

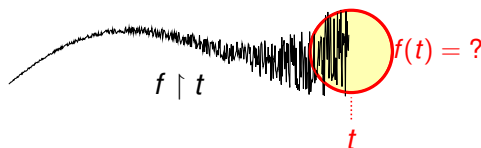
Then there is a strategy by which all will be totally correct except at most one mathematician.

The nearly perfect prediction theorem

Like stock-market traders, we aim to predict the current and future values of an unknown function based on the observed history of prior values.

The nearly perfect prediction theorem

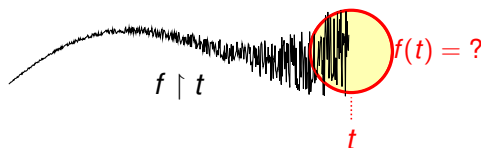
Like stock-market traders, we aim to predict the current and future values of an unknown function based on the observed history of prior values.



The true function $f : \mathbb{R} \rightarrow \mathbb{R}$ is revealed only gradually.

The nearly perfect prediction theorem

Like stock-market traders, we aim to predict the current and future values of an unknown function based on the observed history of prior values.

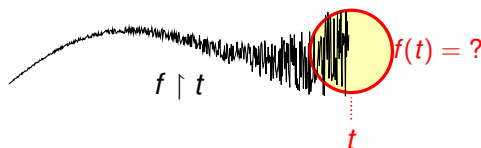


The true function $f : \mathbb{R} \rightarrow \mathbb{R}$ is revealed only gradually.

If the function is continuous, of course, we should predict $f(t) = \lim_{x \rightarrow t} f(x)$.

The nearly perfect prediction theorem

Like stock-market traders, we aim to predict the current and future values of an unknown function based on the observed history of prior values.



The true function $f : \mathbb{R} \rightarrow \mathbb{R}$ is revealed only gradually.

If the function is continuous, of course, we should predict $f(t) = \lim_{x \rightarrow t} f(x)$.

But the function might not be continuous.

Prediction strategies

A *prediction strategy* P looks at the history of prior values and specifies a prediction for the current value.

Prediction strategies

A *prediction strategy* P looks at the history of prior values and specifies a prediction for the current value.

Historical data

$f \upharpoonright t$

$\mapsto$

Predicted value at t

$P(f \upharpoonright t).$

Prediction strategies

A *prediction strategy* P looks at the history of prior values and specifies a prediction for the current value.

Historical data

$f \upharpoonright t$

$\mapsto$

Predicted value at t

$P(f \upharpoonright t).$

The prediction is correct for f at time t if indeed $P(f \upharpoonright t) = f(t)$.

Prediction strategies

A *prediction strategy* P looks at the history of prior values and specifies a prediction for the current value.

Historical data

$f \upharpoonright t$

$\mapsto$

Predicted value at t

$P(f \upharpoonright t).$

The prediction is correct for f at time t if indeed $P(f \upharpoonright t) = f(t)$.

Of course, the prediction is very likely wrong, right? The actual value $f(t)$ could be anything at all.

It seems hopeless to have a prediction strategy that works with any degree of reliability.

One might expect that for any given prediction strategy P , there is a function f for which all the predictions are wrong.

The nearly perfect prediction theorem

Nevertheless, there is a nearly perfect prediction strategy!

The nearly perfect prediction theorem

Nevertheless, there is a nearly perfect prediction strategy!

Nearly perfect prediction theorem, Hardin & Taylor [HT08]

There is a prediction strategy P , such that for every function f the predictions are almost always correct:

$$f(t) = P(f \upharpoonright t).$$

Predictions are correct for f except on a measure-zero set, a countable nowhere dense set. Furthermore, predictions are fully correct on arbitrarily long finite intervals.

The nearly perfect prediction theorem

Nevertheless, there is a nearly perfect prediction strategy!

Nearly perfect prediction theorem, Hardin & Taylor [HT08]

There is a prediction strategy P , such that for every function f the predictions are almost always correct:

$$f(t) = P(f \upharpoonright t).$$

Predictions are correct for f except on a measure-zero set, a countable nowhere dense set. Furthermore, predictions are fully correct on arbitrarily long finite intervals.

It is incredible.

Defining the prediction strategy

By the axiom of choice, we may place a well-order $\trianglelefteq$ on the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Defining the prediction strategy

By the axiom of choice, we may place a well-order $\trianglelefteq$ on the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Given history $f \upharpoonright t$, look at all possible continuations.

Let f^* be the $\trianglelefteq$ -least function that agrees with observation $f \upharpoonright t$.

Defining the prediction strategy

By the axiom of choice, we may place a well-order $\trianglelefteq$ on the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Given history $f \upharpoonright t$, look at all possible continuations.

Let f^* be the $\trianglelefteq$ -least function that agrees with observation $f \upharpoonright t$.

Define the prediction function $P(f \upharpoonright t) = f^*(t)$.

In short, we predict that the function will agree with the least function that stands in agreement with the observed data.

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice $s < t$ implies $f_s^* \sqsubseteq f_t^*$, where these are the functions used in prediction.

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice $s < t$ implies $f_s^* \sqsubseteq f_t^*$, where these are the functions used in prediction.

Therefore, if the prediction is wrong at s , then $f_s^* \triangleleft f_t^*$.

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice $s < t$ implies $f_s^* \sqsubseteq f_t^*$, where these are the functions used in prediction.

Therefore, if the prediction is wrong at s , then $f_s^* \triangleleft f_t^*$.

So incorrect prediction points are well-ordered as real numbers.

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice $s < t$ implies $f_s^* \leq f_t^*$, where these are the functions used in prediction.

Therefore, if the prediction is wrong at s , then $f_s^* < f_t^*$.

So incorrect prediction points are well-ordered as real numbers.

But every well-ordered set in $\langle \mathbb{R}, < \rangle$ is countable & nowhere dense.

So the predictions are almost always correct!

The predictions are almost always correct

Suppose the true function is $f : \mathbb{R} \rightarrow \mathbb{R}$.

Notice $s < t$ implies $f_s^* \trianglelefteq f_t^*$, where these are the functions used in prediction.

Therefore, if the prediction is wrong at s , then $f_s^* \triangleleft f_t^*$.

So incorrect prediction points are well-ordered as real numbers.

But every well-ordered set in $\langle \mathbb{R}, < \rangle$ is countable & nowhere dense.

So the predictions are almost always correct!

Further, every well-ordered set in $\mathbb{R}$ has a least element. So there is an absolutely first wrong prediction, and thus the predictions are totally correct on an interval $(-\infty, s)$.

Predicting the future

In fact, the same prediction method can be used to predict the future values.

Predicting the future

In fact, the same prediction method can be used to predict the future values.

Given observed data $f \upharpoonright t$, we find the $\trianglelefteq$ -least function f^* standing in agreement with it.

For every f and almost every t , these predictions will be correct for a short time into the future:

$$\exists \varepsilon > 0 \quad \forall x < t + \varepsilon \quad f(x) = f_t^*(x).$$

The point is that the incorrect predictions about this are well-ordered as real numbers.

Predicting the future

In fact, the same prediction method can be used to predict the future values.

Given observed data $f \upharpoonright t$, we find the $\trianglelefteq$ -least function f^* standing in agreement with it.

For every f and almost every t , these predictions will be correct for a short time into the future:

$$\exists \varepsilon > 0 \quad \forall x < t + \varepsilon \quad f(x) = f_t^*(x).$$

The point is that the incorrect predictions about this are well-ordered as real numbers.

I should like to emphasize that the future-prediction result is interesting even for continuous functions.

Nonconstructive strategy

The proof of the perfect prediction theorem is nonconstructive.

We have proven that as a matter of mathematical ontology, there is a nearly perfect prediction strategy.

But we cannot say much in detail about its predictions.

Nonconstructive strategy

The proof of the perfect prediction theorem is nonconstructive.

We have proven that as a matter of mathematical ontology, there is a nearly perfect prediction strategy.

But we cannot say much in detail about its predictions.

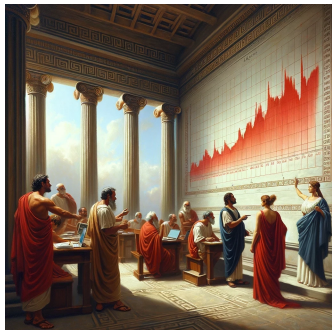
You had best keep your day job.

Existence in the platonic realm

Nevertheless, to my way of thinking, even the pure existence of a nearly perfect prediction strategy is incredible.

Existence in the platonic realm

Nevertheless, to my way of thinking, even the pure existence of a nearly perfect prediction strategy is incredible.



I am confident that the iridescent mathematical stock traders of the platonic realm are truly making bank.

Thank you.

Slides and articles available on <http://jdh.hamkins.org>.

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame