Mathematicians disagree on the essential structure of the complex numbers

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[Ham24] Joel David Hamkins, "Mathematicians disagree on the essential structure of the complex numbers," Infinitely More, November 2024.

https://www.infinitelymore.xyz/p/complex-numbers-essential-structure.

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- Topological structure, a manifold, metric space
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What structure is fundamental?

Mathematicians do not agree—vehement disagreement.

Brief review: How to construct $\mathbb{C}$

We have a variety of methods for constructing the complex number field.

- Complex plane. Hamilton 1833 defines operations on points (a, b) in the real plane.
- Formal sums. Present complex numbers as formal sums $a + bi$ with $a, b \in \mathbb{R}$, subject to $i^2 = -1$.
- Quotient ring. Define $\mathbb{C}$ as the quotient field $\mathbb{R}[x]/(x^2 + 1)$ by the ideal $(x^2 + 1)$. Every polynomial in the quotient can be represented as $a + bx$, where $a, b \in \mathbb{R}$.

Four natural conceptions of $\mathbb{C}$

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- (Analytic) The complex field $\mathbb{C}$ over $\mathbb{R}$
- (Smooth) The topological complex field
- (Rigid) The complex plane
- (Algebraic) The complex field

Four natural conceptions of $\mathbb{C}$

Four natural perspectives on the essential structure of the complex numbers:

- (Analytic) The complex field $\mathbb{C}$ over $\mathbb{R}$
- (Smooth) The topological complex field
- (Rigid) The complex plane
- (Algebraic) The complex field

Two of these, it turns out, arise from the same underlying structural conception. Do you see which two?

So ultimately, we shall have three conceptions.

The analytic conception

Many mathematicians conceive of the complex numbers $\mathbb{C}$ as a field over $\mathbb{R}$, the algebraic closure of $\mathbb{R}$—a degree two algebraic extension.

The perspective is that we have $\mathbb{R}$ already and view $\mathbb{C}$ as built over it by adjoining a square root of -1.

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The perspective is that we have $\mathbb{R}$ already and view $\mathbb{C}$ as built over it by adjoining a square root of -1.

On this view $\mathbb{R}$ is a distinguished subfield of $\mathbb{C}$—it is part of the fundamental underlying structure.

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Since $i^2 = -1$, the imaginary unit i is commonly described as “the square root of -1.”

But of course, there are two such roots.

After all, the negation $-i$ is also a square root of -1:

$$(-i) \cdot (-i) = (-1)^2 \cdot i^2 = i^2 = -1.$$

So that phrase does not serve as a defining property of i.

Indiscernibility of i and $-i$ in $\mathbb{C}$

Can we tell i and $-i$ apart?

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Can we tell i and $-i$ apart?

No, not in the complex field, using only the field structure over $\mathbb{R}$. The numbers i and $-i$ are indiscernible as complex numbers with respect to the algebraic structure of $\mathbb{C}$—any property that i has in the complex field $\mathbb{C}$ over $\mathbb{R}$ will also hold of $-i$.

Complex conjugation is an automorphism of that structure.

$$a + bi \quad \mapsto \quad a - bi$$

There is a fundamental symmetry between i and $-i$.

Complex conjugation as a fundamental symmetry

The analytic conception is thus embodied in the structure

$$\langle \mathbb{C}, +, \cdot, 0, 1, \mathbb{R} \rangle.$$

We have the field structure and the distinguished subfield $\mathbb{R}$.

Symmetry between i and $-i$ is a fundamental symmetry.

Complex conjugation is the only nontrivial automorphism of $\mathbb{C}$ over $\mathbb{R}$.

The smooth conception

The *smooth* conception of $\mathbb{C}$, in contrast, emphasizes that the complex field carries also its topological structure.

The field operations $+$ and $\cdot$ are continuous.

Complex conjugation preserves the topology, so the smooth conception respects the symmetry between *i* and $-i$.

Smooth conception subsumes the analytic conception

In the smooth conception, the real subfield $\mathbb{R}$ is simply the closure of the rationals $\mathbb{Q}$, which are inherent to the field structure.

So $\mathbb{R}$ is distinguished on the smooth conception. Therefore, the smooth conception subsumes the analytic conception.

Smooth = Analytic

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Conversely, with $\mathbb{R}$ distinguished in $\mathbb{C}$, we can define the real-part operator, and this is enough to define the complex norm $|z| = \sqrt{a^2 + b^2}$, without needing to distinguish between i and $-i$.

With the norm, we can define the metric, and so we get the topology as part of this conception.

So the smooth and the analytic conception of $\mathbb{C}$ have the same underlying structure.

Automorphisms for smooth conception

Automorphism of the complex numbers under the smooth conception should respect the topology.

But any continuous automorphism of $\mathbb{C}$ must fix every real number, since it must fix the rational numbers.

It follows that the only continuous automorphisms of $\mathbb{C}$ are the identity automorphism and complex conjugation, which are indeed homeomorphisms of the topology.

The rigid conception

The *rigid* conception of $\mathbb{C}$, which could also be called the *complex plane* conception, we incorporate the full coordinate structure of the complex plane with the field structure.

We consider the complex numbers as the structure $\langle \mathbb{C}, +, \cdot, 0, 1, \mathrm{Re}, \mathrm{Im} \rangle$.

Equivalently, we consider the complex numbers as $\langle \mathbb{C}, +, \cdot, 0, 1, \mathbb{R}, i \rangle$.

This is a structure with no nontrivial automorphisms at all.

The rigid conception breaks the symmetry between i and $-i$.

The algebraic conception

Finally, many mathematicians find it natural to regard the complex numbers at bottom as a field, that is, endowed with its field operations as the only essential structure.

The conception regards the complex numbers as the structure:

$$\langle \mathbb{C}, +, \cdot, 0, 1 \rangle.$$

In particular, this conception respects the fundamental symmetry between i and $-i$.

Categorical account

The complex number field admits a fully categorical characterization.

Namely, $\mathbb{C}$ is the unique algebraically closed field of characteristic zero and size continuum.

Equivalently, the complex field arises as the unique algebraically closed field of transcendence degree continuum over the rational field $\mathbb{Q}$.

Indeed, the theory ACF_0 is categorical in every uncountable power, since an algebraically closed field of characteristic zero is determined by its transcendence degree over the prime subfield $\mathbb{Q}$.

Wild automorphisms

On the algebraic conception of the complex numbers, a spectacular symmetry phenomenon emerges—we have a staggering complexity of automorphisms.

Not just swapping i and $-i$, but there are automorphisms of $\mathbb{C}$ that swap $\sqrt{2}$ and $-\sqrt{2}$, and others permute the cube roots of 5, mapping the real root $\sqrt[3]{5}$ to one of the two nonreal roots, while also independently moving transcendental numbers around.

We can move π to e with an automorphism of $\mathbb{C}$.

The general fact is that every complex number, except the rational numbers, is moved by some automorphism of $\mathbb{C}$.

Wild automorphisms of $\mathbb{C}$

There are $2^{2^{\aleph_0}}$ many such “wild” automorphisms of $\mathbb{C}$.

The automorphisms do not generally respect the real line—on the contrary they tear it to shreds, scattering it densely over the complex plane.

The wild automorphisms do not ever fall under a simple description or formula—one rather constructs them in a transfinite process using the axom of choice.

$\mathbb{R}$ not definable in $\mathbb{C}$

There are more than continuum many distinct copies of $\mathbb{R}$ in $\mathbb{C}$—all the various automorphic copies of $\mathbb{R}$ scattered in the complex plane.

Every such copy is a complete ordered field, and $\mathbb{C}$ arises over each of them as the algebraic closure, simply by adjoining i.

We can't pick out the familiar copy of $\mathbb{R}$ in $\mathbb{C}$ using only purely algebraic features.

Subfields of $\mathbb{C}$ isomorphic to $\mathbb{C}$

In fact, there are numerous proper subfields of $\mathbb{C}$ that are isomorphic to $\mathbb{C}$.

To see this, consider a transcendental extension $\mathbb{C}(x)$ and then take the algebraic closure $\overline{\mathbb{C}(x)}$.

This is an algebraically closed field of characteristic zero and size continuum, and hence it is isomorphic to $\mathbb{C}$.

The original copy of $\mathbb{C}$ is a proper subfield of $\overline{\mathbb{C}(x)}$.

Mathematicians disagree

It turns out that mathematicians do not agree on what $\mathbb{C}$ is.

They disagree on what is the essential structure.

Poll results

I ran an unscientific poll on Twitter that revealed a dispersed range of views in the mathematical/philosophical community for how to conceive of the complex numbers.

Joel David Hamkins
@JDHamkins

How do you think of the complex numbers? What is the core structure?
1. The complex numbers, at bottom, are an algebraically closed field $\langle \mathbb{C},+,\cdot,0,1\rangle$.
2. The complex numbers are the algebraic closure of the reals, a distinguished subfield $\langle \mathbb{C},+,\cdot,0,1,\mathbb{R}\rangle$.
3. The complex numbers are an algebraically closed field with a distinguished real coordinate structure $\langle \mathbb{C},+,\cdot,0,1,\mathrm{Re},\mathrm{Im}\rangle$.

As a field	27.9%
As algebraic closure of $\mathbb{R}$	32.4%
With coordinate structure	**39.7%**

1,866 votes · Final results

7:24 PM · Oct 28, 2024 · **44.5K** Views

Disagreement

Despite the variation in views, many mathematicians are passionate about their favored answer, viewing those with the opposing view as making a fundamental mistake.

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Despite the variation in views, many mathematicians are passionate about their favored answer, viewing those with the opposing view as making a fundamental mistake.

For example, one respondent expressed the view that the algebraic conception is “definitely wrong.”

Another says, “the coordinate view of [the complex plane] is what lives in my spine.”

Mathematician Barbara Fantechi (algebraic geometry) states, “I was taught that choosing a square root of (-1) is wrong.”

More disagreement

Mathematician Daniel Litt (algebraic geometry, number theory) mentions that the rigid conception "is tantamount to choosing a square root of -1, I guess, which I think of as immoral."

He adds that "At the very least making such a choice often leads to confusion later on."

The view is that somehow it is a kind of mathematical sin to break the symmetry between i and $-i$ in $\mathbb{C}$ over $\mathbb{R}$ and thus wrong to adopt the rigid perspective of the complex plane conception of $\mathbb{C}$.

Nevertheless, mathematician Rogier Brussee responds that, "Immoral sure, but, well, you know, the snake and the apple things; it can be a bloody convenient immorality at times."

Set-theoretic indiscernibility

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Theorem (Hamkins)

If ZFC is consistent, then there is a model of ZFC that has a definable complete ordered field $\mathbb{R}$ with a definable algebraic closure $\mathbb{C}$, such that the two square roots of -1 in $\mathbb{C}$ are set-theoretically indiscernible, even with ordinal parameters.

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The aims are in tension, since we want $\mathbb{C}$ to be definable in V, hence the set $\{\, i, -i \,\}$ is definable, but the individual elements i and $-i$ are not definable in the set-theoretic background.

Not always possible in ZFC, since pointwise definable models.

Proof idea: By forcing, create Groszek-Laver pair $\{\, i, j \,\}$, definable but no OD element. Form quotient $\mathbb{R}[i, j]/(i^2 + 1, j^2 + 1, i + j)$, which makes i, j the roots of -1, but neither is definable.

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Reason: only countably many 1-types in $\mathbb{C}$, even with parameters. But uncountably many in $\mathbb{R}$.

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This situation surprises many mathematicians, since $\mathbb{C}$ is easily interpretable in $\mathbb{R}$ via pairs, and mathematicians often presume the converse direction should be even easier. But it isn't.

Structuralism—two parallel threads

A philosophical thread

Growing out of Benacerraf's influential work. Concerned with core philosophical issues, mathematical ontology, the problem of reference to and interaction with abstract objects. Many varieties: abstract, ante-rem, eliminative, modal,...

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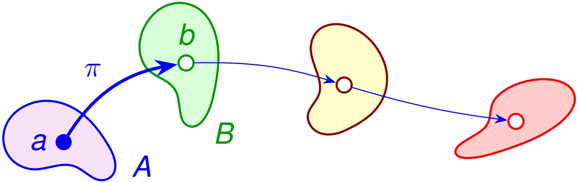
A mathematical thread

Tracing back to the categoricity results of Dedekind, Huntington. Currently pervasive in mathematics: treat mathematical objects only up to isomorphism. Structuralist imperative.

Mathematicians talk at length about structuralism, but almost never mention Benacerraf or the other philosophical literature.

Structural role

Object a in structure A plays same *structural role* as object b in structure B exactly when there is an isomorphism of A with B carrying a to b.



The *isomorphism orbit* of an object in a structure is the resulting equivalence class.

Structural role

According to Shapiro [Sha96; Sha97] the slogan of structuralism is that "mathematics is the science of structure."

Leads to ante-rem or abstract structuralism.

Shapiro advanced the view that what mathematical objects are, at bottom, is the structural roles that those objects play in a mathematical system.

This is an account that aims to tell us the essential nature of mathematical ontology.

Criticism of abstract structuralism

Abstract structuralism is anti-structuralist

To say what the numbers *really* are is to violate the structuralist imperative. It just doesn't matter what mathematical objects really are, even if the answer provided is that they are purely structural. [Ham21, p.35–36].

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A problem with nonrigid structure

The structural role played by i in $\mathbb{C}$ is identical to the structural role played by $-i$ in $\mathbb{C}$, because they are automorphic. Same role, distinct objects. So the objects can be identified with the role.

Nonrigid structuralist existence

Structuralist reference to our familiar mathematical structures—$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$—is grounded in the categoricity results characterizing these structures.

We don't point at a particular field, kept like the platinum rod in Paris, to serve as the canonical instance. Rather, we just say, "take any complete ordered field." We only care about $\mathbb{R}$ up to isomorphism.

Each of our various conceptions of $\mathbb{C}$ admits such a categorical account.

A problem with nonrigid structures

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All constructions of $\mathbb{C}$ over $\mathbb{R}$ of which I am aware break the symmetry between i and $-i$ during the construction.

- Formal sums $a+bi$ present i as a constant, to be forgotten.
- Quotient construction $\mathbb{R}[x]/(x^2+1)$ in effect names x, which becomes i. Choice between x and $-x$.
- Complex plane (a, b) distinguishes $(0, 1)$.

Rigid structure first

Litt said that it was mathematically immoral to break the symmetry between i and $-i$.

But all constructions of $\mathbb{C}$ do this. We first build a rigid structure, and then forget the extra structure.

Must we be immoral?

Rigidity first

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How could it be otherwise?

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> *We don't start with a naked copy of $\mathbb{C}$ and then seek to impose an orientation on it that will enable us to resolve i from $-i$. Rather, we proceed oppositely: instances of mathematical structures are obtained from richer contexts where the objects were already individuated.* [Ham21, p.46–47]

Furthermore, we can prove in ZF that all structures arise from rigid structures in this way.

Nonrigid structure is not fundamental

Some mathematicians have expressed the symmetric conceptions of $\mathbb{C}$ as the core fundamental notion.

Nonrigid structure is not fundamental

Some mathematicians have expressed the symmetric conceptions of $\mathbb{C}$ as the core fundamental notion.

My objection is that all constructions of $\mathbb{C}$ proceed by first constructing a rigid copy, and then forgetting structure to get the symmetric conceptions.

Therefore, the nonrigid conceptions of $\mathbb{C}$ cannot be fundamental.

Conclusion

We began with the question, how shall we view the complex numbers?

Opinion is split, but many mathematicians prefer the nonrigid conceptions, admitting the fundamental symmetry between i and $-i$.

Nevertheless, all the usual constructions of $\mathbb{C}$ have the rigid conception at their core, with symmetries reintroduced only afterward by forgetting the extra structure used in the construction.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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