On Skolem's paradox and the transitive submodel theorem

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

VRF, Mathematical Institute, Oxford

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Joint work with Timothy Button, UCL.

With thanks also to W. Hugh Woodin, Harvard.

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Thus our notions of countability and uncountability are not absolute.

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Also natural to focus on *transitive* models $\langle W, \in \rangle$, those for which all the actual elements of any set $a \in W$ are also in W.

Transitive models know fully about their sets.

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Note that the submodel $\langle U, \in\rangle$ may not be transitive.

In the nontransitive case, elements of U can be uncountable, even though U is countable.

A better paradox via Mostowski

Given any transitive model $\langle W, \in\rangle \models \text{ZFC}$, apply LS to find countable elementary submodel $U \prec W$.

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Mostowski collapse

Every well-founded extensional structure $\langle U, E \rangle$ is isomorphic to a transitive set via the Mostowski collapse

$$\pi(y) = \{ \pi(x) \mid x \, E \, y \}.$$

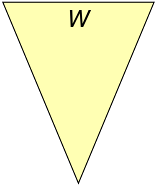
The range $M = \{ \pi(y) \mid y \in U \}$ is transitive, and

$$x \, E \, y \iff \pi(x) \in \pi(y).$$

So $\pi : \langle U, E \rangle \cong \langle M, \in \rangle$.

Countable transitive models

Putting it together:



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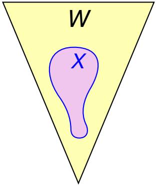

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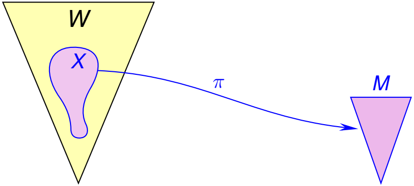

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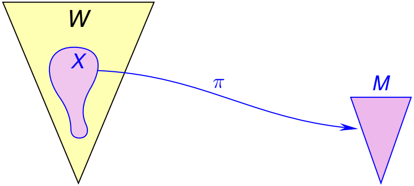

Given transitive $W \models$ ZFC, find a countable $X \prec W$ by Löwenheim-Skolem, and then apply Mostowski to get countable transitive model M.

Conclusion

If there is a transitive model of ZFC, then there is a countable transitive model of ZFC.

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Given Löwenheim-Skolem context, natural to take him as intending elementary submodel. Or just same theory?

From the Skolem paradox context, it is clear that he wants at least submodel $\models$ ZF.

Stronger form in SEP

SEP entry on Skolem's paradox, Bays 2014

The *Downward Löwenheim-Skolem Theorem* says that if *N* is a model of (infinite) cardinality κ and if λ is an infinite cardinal smaller than κ, then *N* has a submodel of cardinality λ which satisfies exactly the same sentences as *N* itself does.

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Finally, the *Transitive Submodel Theorem* strengthens the downward Löwenheim-Skolem theorem by saying that if our initial *N* happens to be a so-called transitive model for the language of set theory, then the submodel generated by the downward theorem can also be chosen to be transitive.

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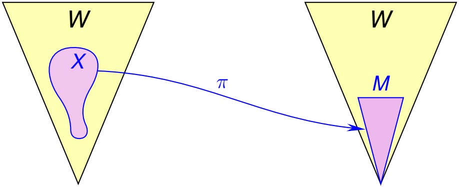
Finally, the *Transitive Submodel Theorem* strengthens the downward Löwenheim-Skolem theorem by saying that if our initial *N* happens to be a so-called transitive model for the language of set theory, then the submodel generated by the downward theorem can also be chosen to be transitive.

Seems to assert only the same theory for the submodel, rather than being an elementary submodel.

The intended proof

Given transitive $W \models \text{ZFC}$, get countable $X \prec W$ by Löwenheim-Skolem, and then apply the Mostowski collapse:



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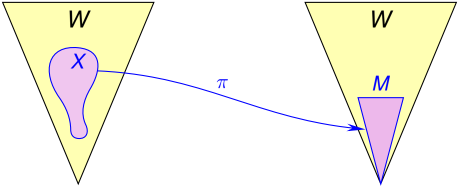

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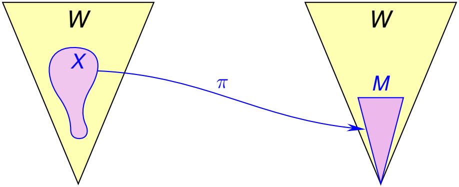

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Indeed, we would expect an elementary embedding $j : M \to W$, since $j = \pi^{-1} : M \cong X \prec W$.

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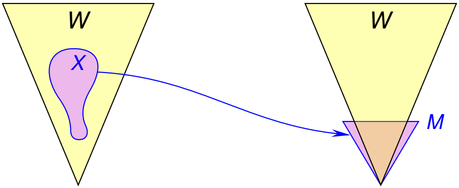

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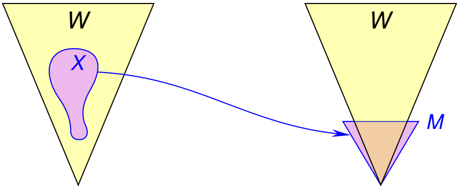

Why should it stay as a submodel of W? I couldn't see why.

It was called the Transitive Submodel Theorem, but is it actually a theorem? We don't seem to have much reason to expect a transitive *submodel* $M \subseteq W$ with the same theory.

A spectrum of Skolem reflection principles

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Transitive submodel of ZFC (weak reading of Benacerraf)

Every transitive model of set theory $W \models \text{ZFC}$ has a countable transitive submodel $M \subseteq W$ with $M \models \text{ZFC}$.

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Transitive submodel theorem (SEP)

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Transitive elementary submodel theorem (strong Benacerraf)

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Sorting it out

I should like to sort out the various principles, identify which are theorems, which are not, which are independent of ZFC, and investigate the consistency strengths of the principles and their negations.

Provable instances

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- In particular, for models W of height ω_1
- For models $W \models \text{ZFC}$ for which $\text{Th}(W) \in W$.

Let me go through some of these arguments now.

Models of $V = L$

Theorem

Every uncountable transitive $W \models \text{ZFC} + V = L$ has a countable transitive $M \subseteq W$ with same theory. Indeed, we can have $M \in W$ and elementary $j : M \to W$.

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This argument essentially proves the existence of the Shepherdson-Cohen model, the least $L_\alpha \models$ ZFC. This model is a submodel of all transitive models of ZF.

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If $W \models \mathrm{ZFC}$ is transitive and the height of W is an ordinal with uncountable cofinality, then there is transitive $M \subseteq W$ with same theory and indeed $M \in W$ with elementary $j : M \to W$.

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Inside W, find M via LS+Mostowski, so $\exists j : M \to W_\kappa \prec W$. □

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Indeed, since $W_\kappa \prec W$, all definable elements are below κ.

In general, in ZFC we cannot say, "let κ be the supremum of the first Σ_n-correct cardinal for $n < \omega$."

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Definable cuts work in models of PA also.

Models arising from large cardinals

Corollary

If κ is an inaccessible cardinal, then V_κ has a countable transitive submodel $M \subseteq V_\kappa$ with the same theory, and indeed, there is such an M with an elementary embedding $j : M \to V_\kappa$ with both $M, j \in V_\kappa$.

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Does not (yet) cover worldly cardinals, $V_\kappa \models$ ZFC, since these can be singular.

Models containing their own theory as an element

Theorem

Suppose $W \models \mathrm{ZFC}$ *is transitive and contains its own theory as an element* $\mathrm{Th}(W) \in W$*. Then there is a countable transitive submodel* $M \subseteq W$ *with the same theory. For uncountable* W*, can find* $M \in W$*.*

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Hence true in $L_{\omega_1}[T]$ by Shoenfield absoluteness theorem.

But $L_{\omega_1}[T] \subseteq W$, hence true in W. □

Countable transitive submodel theorem under $V = L$

Corollary

If $V = L$, then the transitive submodel theorem is true—every transitive $W \models \text{ZFC}$ has countable transitive submodel $M \subseteq W$ with same theory.

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Only need that $\mathbb{R} \subseteq L$. Then every real is in L_{ω_1}.

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So there is countable transitive $M \subseteq W$ with same theory. □

Strong Benacerraf principle

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Can find a club of elementary substructures $(V_\kappa)^W \prec W$, which each serve as an M. □

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And yet the conclusion was true in many cases.

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Ultimately, however, it turns out the principle is overstated—it is not a theorem of ZFC.

Let me present a counterexample.

Main counterexample

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Both assumptions are necessary.

- If every real is constructible, the transitive submodel theorem holds.
- If transitive models have height at most ω_1, then strong Benacerraf holds.

Main Lemma

If $L_\theta[x]$ is an uncountable transitive model of ZFC, containing x as an element, then the following are equivalent.

1. There is a countable transitive submodel $M \subseteq L_\theta[x]$ with the same theory as $L_\theta[x]$.
2. There is a countable transitive model $M \in L_\theta[x]$ with the same theory as $L_\theta[x]$.
3. The theory of $L_\theta[x]$ is an element of $L_\theta[x]$.

Proof.

Key step, for $(1 \to 2)$, if $M \subseteq L_\theta[x]$ has same theory, then $M = L_\alpha[\bar{x}]$ for some $\bar{x} \in M \subseteq W$ and countable α, so $M \in L_\theta[x]$. □

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So containing own theory is necessary and sufficient for the transitive submodel theorem for models of form $L[x]$.

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Follows that $\lambda = \sup_n \lambda_n$. Singular in V, regular in L_θ.

Counterexample proof outline, continued

Similar analysis shows that for any $\leq \omega_1$-distributive forcing notion $\mathbb{P} \in L_\theta$, we can construct in V an L_θ-generic $G \subseteq \mathbb{P}$.

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So $\mathrm{Th}(W) \notin W$.

Since W has form $L_\theta[x]$, this is a counterexample to the countable transitive submodel theorem. $\square$

Necessary and sufficient characterization

Since the hypotheses were necessary, we may deduce:

Corollary

The transitive submodel principle holds if and only if every real is constructible or all transitive models of ZFC *have height at most* ω_1.

Large cardinals refute the principle

All the usual large cardinals imply that there are transitive models of ZFC taller than ω_1.

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And all moderately large large cardinals imply that there are nonconstructible reals, such as $0^\sharp$.

So all such large cardinals directly refute the countable transitive submodel principle.

Thus, the countable transitive submodel principle can be seen as an anti-large-cardinal principle

Consistency strength

It follows that the transitive submodel proposition is equiconsistent with ZFC.

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Alternative proof: the transitive submodel principle follows from $V = L$, which is equiconsistent with ZFC.

Strength of negated principle

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The failure directly implies the existence of an uncountable transitive model of ZFC.

Conversely, if W is an uncountable transitive model, then if height of W is $> \omega_1$, we get the counterexample in a Cohen extension.

And if height of W is exactly ω_1, then ω_1 is inaccessible in L, and so there are counterexamples in L.

Submodels versus elements

A further subtle matter.

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Is this necessary?

Submodel but no element

The answer is no.

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Theorem

If there is a nonconstructible real number and there is a transitive model of ZFC *with height exceeding* ω_1*, then there is an uncountable transitive model* W *of* ZFC *that admits a countable transitive submodel* $M \subseteq W$ *with the same theory, but admits no such model* M *as an element* $M \in W$*.*

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The proof is a class-forcing version of the main counterexample argument. We arrange a submodel $M \subseteq W$, but $\mathrm{Th}(W) \notin W$.

Submodel versus embedding

There are many similar questions regarding the distinction between having $M \subseteq W$ with same theory and having $M \subseteq W$ with $j : M \to W$.

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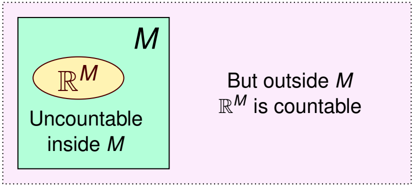
This is part of ongoing work.

Addendum on the iterated Skolem paradox

Skolem observed that a model of set theory M can be wrong in its judgment that a set is uncountable.

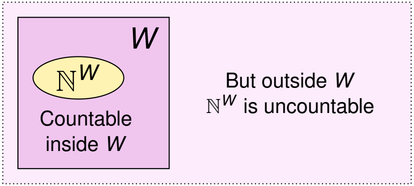


If M is any countable model of ZFC, then the set of real numbers $\mathbb{R}^M$ is uncountable from M's perspective, but countable outside M.

Judgments that a set is countable or uncountable can thus depend on the set-theoretic background in which they are made.

Reverse Skolem paradox

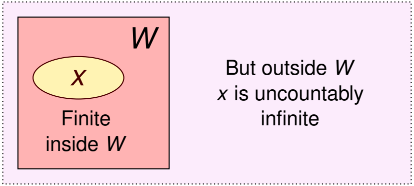
We can also arrange the reverse situation, where a set is countable inside a model W, but uncountable outside.



For example, take the theory ZFC with uncountably many constants n_α, asserting they are pairwise distinct natural numbers. By compactness, there is a model.

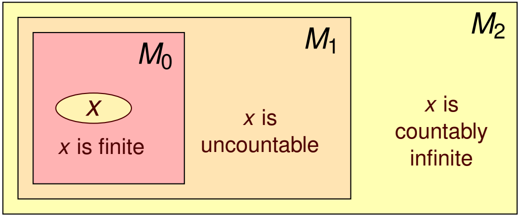
Extreme reverse Skolem paradox

By pushing this harder, we can have all $n_\alpha < x$ for some natural number x in W.



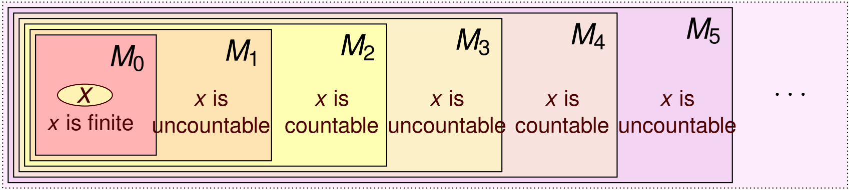
Iterated nonabsoluteness

There are models $M_0 \subseteq M_1 \subseteq M_2$ of ZFC viewing a set x as finite, then uncountable, and finally countably infinite:



Indefinitely iterated indefiniteness

Indeed, we can extend the pattern of indefiniteness indefinitely.



Finiteness is downwards absolute from any metatheoretic context to the models available there.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

VRF, Mathematical Intitute
University of Oxford

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