Infinite-time computable analogues of the universal algorithm

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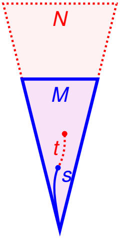

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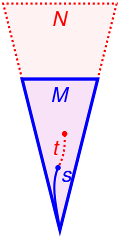

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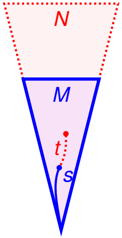

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Proof proceeds by a highly self-referential algorithm, "the petulant child."

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(The extenson property of the universal algorithm is more.)

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- Stage n succeeds, if there is a proof from fragment PA_{k_n}, with k_n smaller than all earlier k_i, of a statement of the form

  "it is not the case that e has exactly n stages and releases s at stage n,"

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  where s is a finite list of explicit numerical terms.
- In this case, release s at stage n. Proceed to next stage.

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Succinctly:

The program e enumerates s at stage n, if it finds proof, in a strictly smaller fragment of PA each time, that it does not do so as its last stage.

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Note

Use Kleene recursion theorem to find e, solving the circular definition.

Finiteness

Observation

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Proof.

The fragments PA_{k_n} must descend, and so there can be at most finitely many successful stages.

Thus, the sequence enumerated by e will be finite.

And PA can undertake this argument. □

Empty in the standard model

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has complexity Σ^0_2. For standard k, the Mostowski reflection theorem shows $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{Tr}_k)$.

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In particular, e enumerates empty sequence in $\mathbb{N}$.

Proving the extension property of e

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Since stage n was not successful, M must think that

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So N end-extends M and thinks e enumerates t. And $N \models \mathrm{PA}$ since k is nonstandard. This proves the theorem. □

Classical consequences

Several classical results in the model theory of arithmetic can be seen as immediate consequences of the universal algorithm.

Let us explore a few examples.

Maximal Σ_1 diagrams

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Proof.

If $M \models \mathrm{PA}$, then there is an unsuccessful stage n, which becomes successful in an end-extension N.

So the assertion “stage n is successful” is a new Σ_1 statement about n true in N, false in M. □

For example, there is a diophantine equation, with coefficients in M, having no solution in M, but it has a solution in N.

Independent Π^0_1 sentences

Corollary (Kripke, Mostowski)

There are infinitely many independent Π^0_1 sentences

$$\eta_0,\ \eta_1,\ \eta_2,\dots$$

Any desired true/false pattern is consistent with PA.

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Proof.

Let $\eta_k =$ "k does not appear on the universal sequence." □

Independent buttons

Corollary "independent buttons"

There are Σ^0_1 sentences

$$\rho_0,\ \rho_1,\ \rho_2,\ \cdots$$

all false in $\mathbb{N}$ and for any $M \models \mathrm{PA}$, any pattern I coded in M, there is end-extension N with

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Proof.

Let $\sigma_k =$ “k is amongst the numbers added at the last stage.” □

Flexible formulas

Corollary (Kripke)

For $n \geq 2$, there is a Σ^0_n formula $\varphi(x)$ that can be made so as to agree with any desired Σ^0_n formula $\psi(x, a)$ in an end-extension.

That is, for any $M \models \mathrm{PA}$ and any such ϕ and $a \in M$, there is an end-extension N satisfying

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For $n \geq 2$, there is a Σ_n^0 formula $\varphi(x)$ that can be made so as to agree with any desired Σ_n^0 formula $\psi(x, a)$ in an end-extension.

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Proof.

Let $\varphi(x) = \Phi(k, x, a)$ where $\langle k, a\rangle$ is last on the universal sequence and Φ is a universal Σ_n^0 formula. □

Arithmetic potentialism

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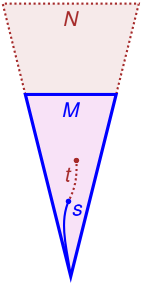
The proof uses the universal algorithm to code arbitrary finite pretrees. At bottom, possibility branches like these trees.

Generalization to Σ_m-elementary extensions

The universal algorithm generalizes to Σ_{m+1}-definable finite sequence, with the universal extension property with respect to Σ_m-elementary end-extensions $M \prec_{\Sigma_m} N$.



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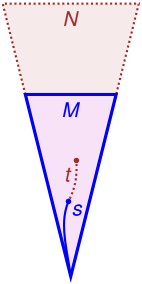

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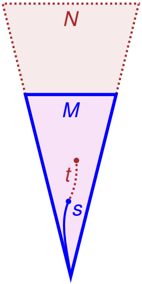

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But the difference now is that Σ_m truth is preserved between M and N.

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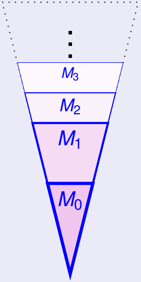

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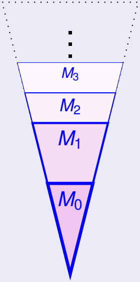

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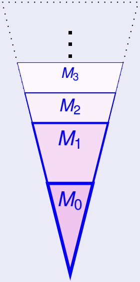

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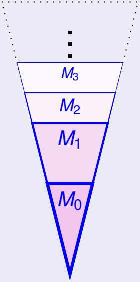

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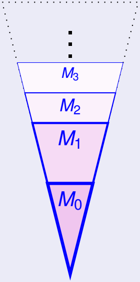

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Can arrange that every element becomes definable. So N is pointwise definable. □

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Proof strategy. Given $M_0 \models \mathrm{PA}$ of size at most continuum, construct a progressively elementary tower

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- Even stages, fully elementary. Create a countable set of points from which previous elements are discernible.
- Odd stages, progressively elementary. Make those points definable.

Ordinal Turing machines

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Programs are finite instruction sets. Tape cells have 0 and 1. At limits, update head position and state with $\liminf$.

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Theorem

There is OTM universal algorithm.

1. Enumerates a finite sequence of ordinals, provably in ZF.
2. In any countable $M \models \text{ZF}$, if sequence is s, then for any $s \subseteq t$ in M there is end-extension $M \subseteq N$ in which the computed sequence is t.

The program runs longer in the extension model and places the desired additional ordinals onto the sequence.

Let's get into it.

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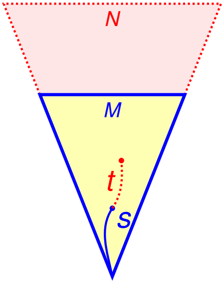

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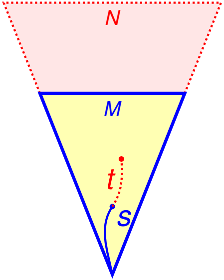

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The definition (complex, sophisticated) essentially looks for stages V_α that have no end-extension adding a next point a, even in any forcing extension, and when found, adds a anyway. “petulant child”

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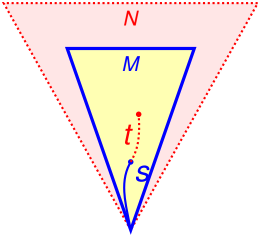

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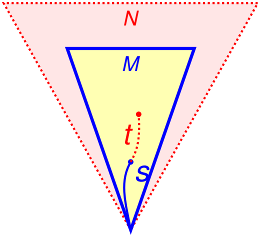

If sequence is s in countable $M \models \text{ZFC}$, then for any desired t, there is an end-extension $N \models \text{ZFC}$ in which the sequence is t.

In fact, can get $N \models \overline{\text{ZFC}}$ for any theory true in some inner model W of M.

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In each case, a certain highly self-referential definable sequence has a universal extension property for extensions of the given type.

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Ultimately, we merge the two processes into a single definition with the desired properties.

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We claim all k_n are nonstandard.

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Since stage n did not succeed, for every countable transitive m in L^M containing earlier m_i and every $a \in m$, the structure $\langle m, \in\rangle^M$ does have a covering end-extension in M to a model making every set in m countable and satisfying $\overline{\mathrm{ZF}}_k +$"object a was placed onto the sequence at stage n, the last successful stage."

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Apply it to a covering set $m = (V_\theta)^{M^+}$, and get a covering end-extension of M, where object a is added as next/last object on sequence, as desired.

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So there must be an extension after all, and this provides an extension of M, as desired.

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Must ensure the witnesses are appropriately bounded in order to prevent interference. But it works. $\square$

This proves the Σ_1-definable universal finite sequence.

OTM universal algorithm

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Theorem

There is OTM universal algorithm.

1. Enumerates a finite sequence of ordinals, provably in ZF.
2. In any countable $M \models \mathrm{ZF}$, if sequence is s, then for any $s \subseteq t$ in M there is end-extension $M \subseteq N$ in which the computed sequence is t.

Key point: OTM computations can search for witnesses of Σ_1 assertions, and so the Σ_1-definable universal sequence is OTM computable.

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I don't yet have a complete version of this argument.

It seems the ω-standard/nonstandard case division will remain fundamental, and so perhaps the current arguments are already achieving it.

Generalization to Σ_m elementary end-extensions

In [Ham24], I have generalized to find a Σ_{m+1} definable sequence of ordinals

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n$$

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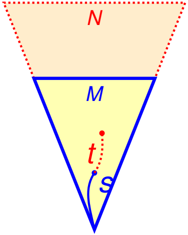

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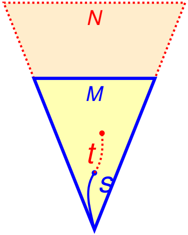

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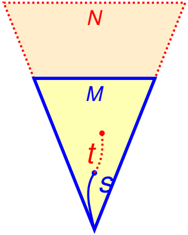

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If $V = \mathrm{HOD}$, can translate this to all objects, not just ordinals.

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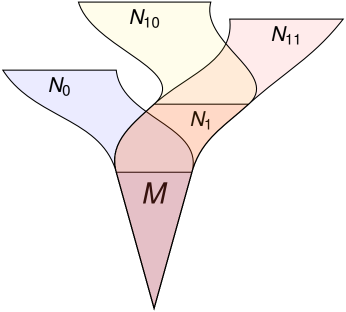
- Modal logic of end-extension potentialism is exactly $\mathrm{S4}$

The tree of top-extensions



Radical-branching potentialism.

Pointwise definable extensions in set theory

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This realizes a certain *resurrection* property: whatever is true in some inner model can become true again in an end-extension, even a pointwise definable end-extension. e.g. Inner models of large cardinals resurrected in end-extensions.

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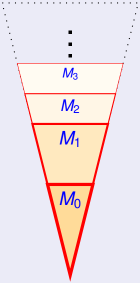

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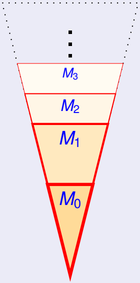

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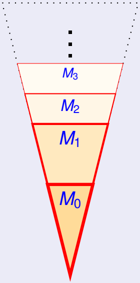

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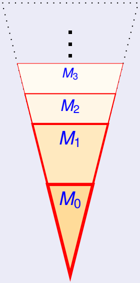

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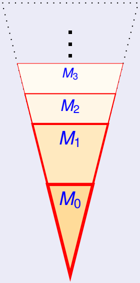

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Can arrange that every element becomes definable. So N is pointwise definable. □

Leibnizian extensions

And similarly, in joint work with myself and Gitman, we've proved that every model of ZFC of size at most continuum has an extension that is Leibnizian.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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O'Hara Professor of Logic
University of Notre Dame

VRF, Mathematical Institute
University of Oxford

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