An exploration of infinite games—infinite Wordle and the Mastermind numbers

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Recursive chess

The Chocolatier's game

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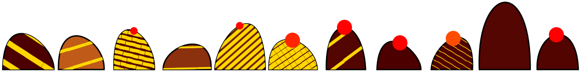

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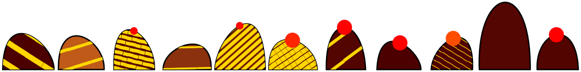

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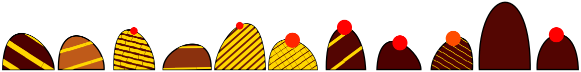

Each round, the *Glutton* selects one and eats it.

Glutton wins, after infinite play, if all chocolates are consumed.

Can the Glutton win?

Chocolates are served, the Glutton eats one.

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Nevertheless, of course, the Glutton can easily win.

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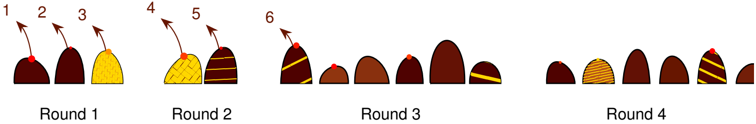

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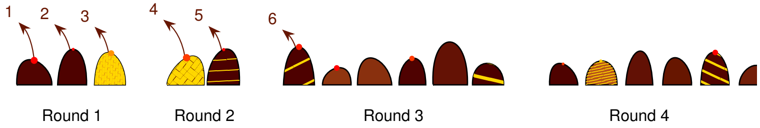

Newly served chocolates are placed at the back of the queue.

But the Glutton eats always from the front.

Every chocolate is finite distance from the front, and so all chocolates will be eaten.

Winning with infinitely many chocolates each round

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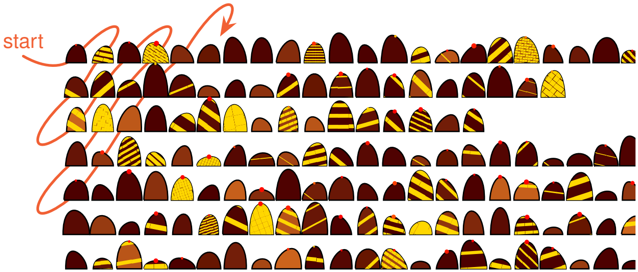
To win, the Glutton imagines placing them on a fresh row, and eats along a winding path.



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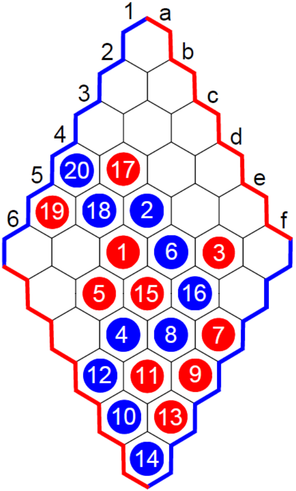
- What if the Chocolatier is uncountably creative?
- In this case, there can be no memory-free winning strategy for the Glutton.
- Nevertheless, assuming the axiom of choice, the Glutton can win while knowing only chocolates on offer + the previously eaten chocolate.

Hex



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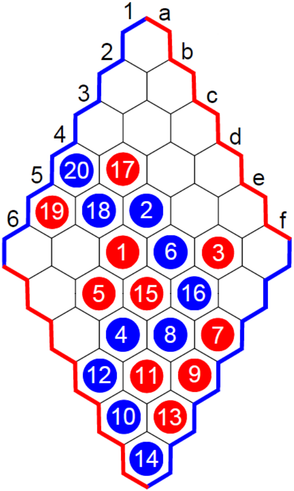

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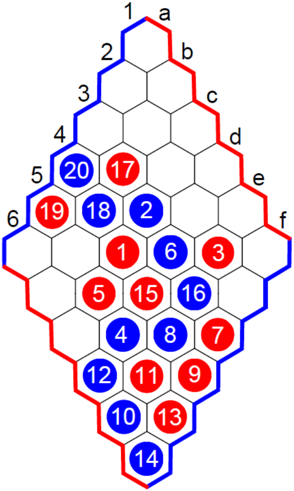

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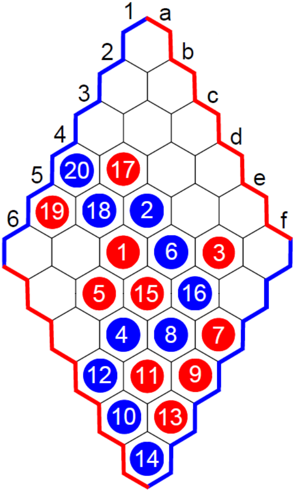

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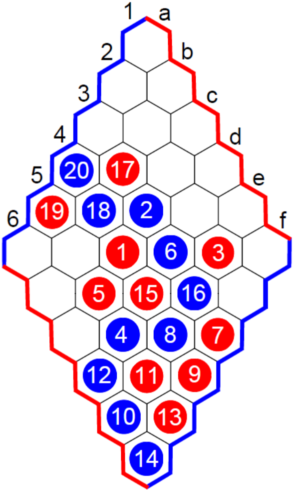

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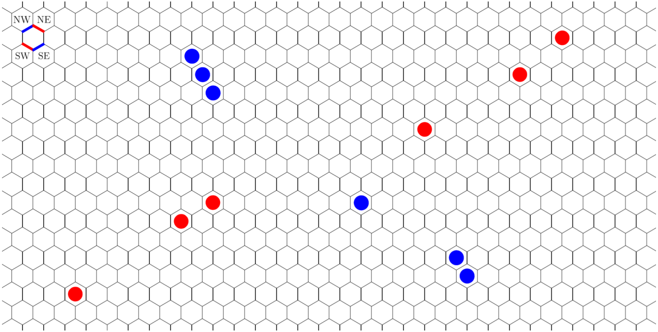
- So first player must have winning strategy

Infinite Hex



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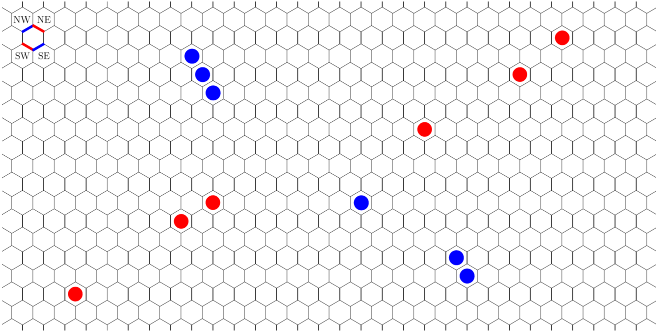

Place stones on infinite hex board.

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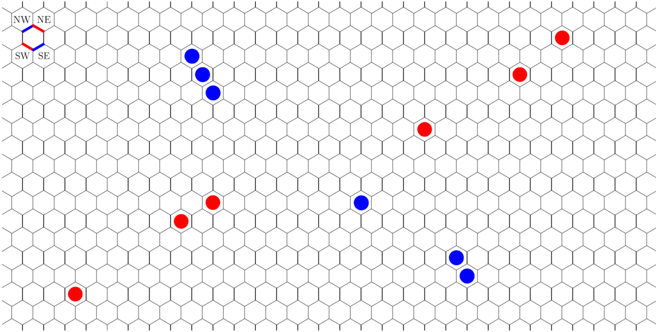

Place stones on infinite hex board.

Each player aims to create a chain connecting opposite corners "at infinity."

Infinite Hex is a draw

Theorem (Hamkins & Leonessi [HL23; Leo21])

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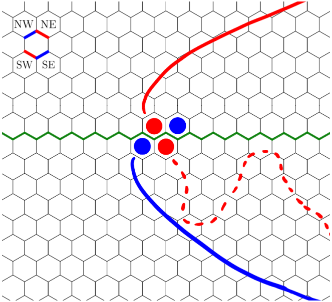

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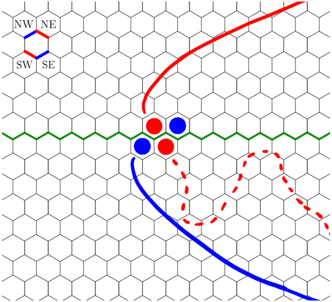

By strategy-stealing, second player cannot have a winning strategy.

But second player has a drawing strategy.

Idea: Blue reflects Red play, with a shift.

Because of the crossing geometry, if Red succeeds at upper right, Blue contains at lower left.

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- Loss avoiding
- De Morgan law
- Ordinal game values

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What is a finite game? Hypergame paradox.

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- A node gets opponent's label if all moves lead to a child node with that label.

The root node will get one label or the other, and whoever it is can win—play to stay on your labels. $\square$

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So second player wins by avoiding the losing nodes. $\square$

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This asserts precisely that player 1 has a winning strategy! $\square$

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Theorem (Clopen determinacy)

In every clopen game, one of the players has a winning strategy.

A game is *clopen* if all plays of the game end in finitely many moves. Back-propogation proof still applies via transfinite recursion.

Open games

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Infinite draughts also is open, since infinitely long play is a draw.

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Alternative proof—transfinite ordinal game values.

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Strong connection with Sprague-Grundy nimbers.

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- If it is the closed player's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet defined.

Some positions may have no value, and this is fine—these are the unvalued positions.

Fundamental observation of open game values

If a position has a value

The open player can win by the value-reducing strategy.

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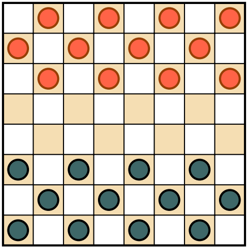
This also amounts to a fourth proof of the fundamental theorem, since finite games are open.

Finite draughts

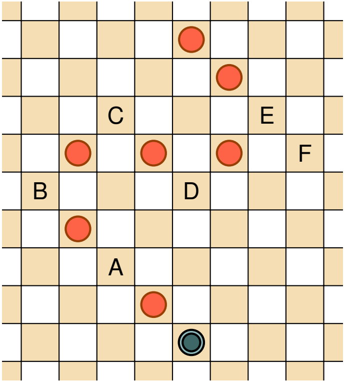
The familiar game of draughts, also known as checkers.



Although often played by children, draughts admits serious advanced play, with draughts grandmasters competing in international tournaments.

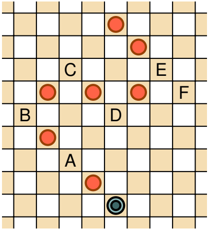
Infinite draughts

Let us endeavor instead to play *infinite* draughts.



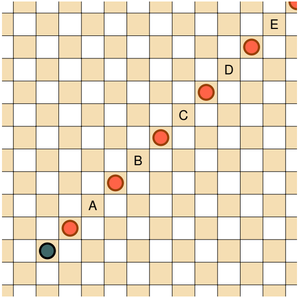
The checkerboard extends without end in every direction.

The rules



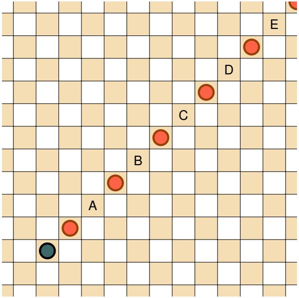
- No standard starting configuration.
- Analyze the game from any given position.
- Pawns and kings.
- Obligatory jumping rule.
- Obligatory iterated jumping.
- Winning condition: you lose when you have no legal move.

Infinite iterations



During an infinite iterated jump, the red pieces are all removed.

Infinite iterations



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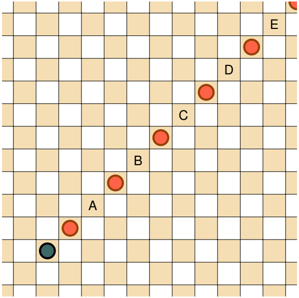

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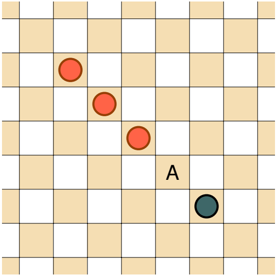
Infinite iterated-jump rule: the jumping piece also is removed.

Game value 2



This position, with Red to play, has game value 2 for Red.

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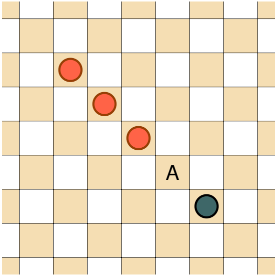

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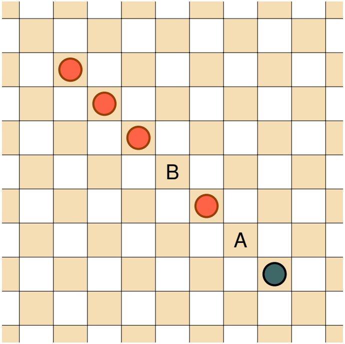
Red can advance leading pawn to A as bait, obligating Black to jump, after which Red recaptures.

Game value 3



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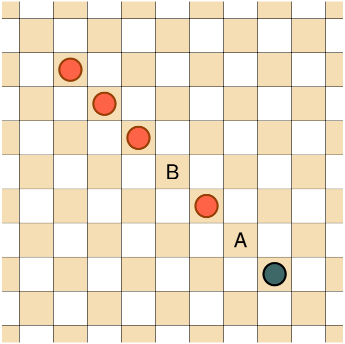
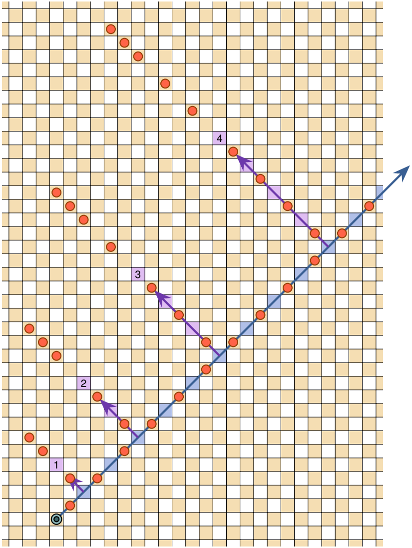


Red to play, with game value 3.

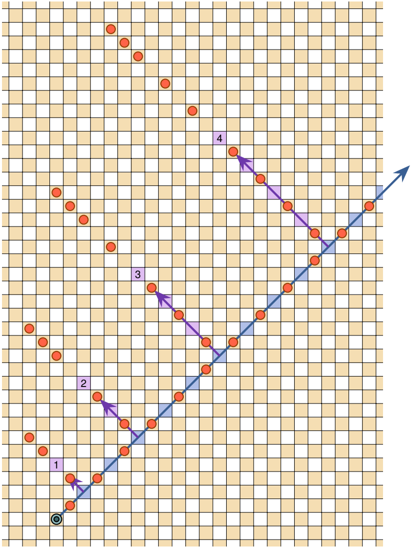
Red advances isolated pawn to A, obligating Black to jump, then advance to B, Black jumps, then Red captures.

Game value ω



Black is obligated to jump.

Game value ω

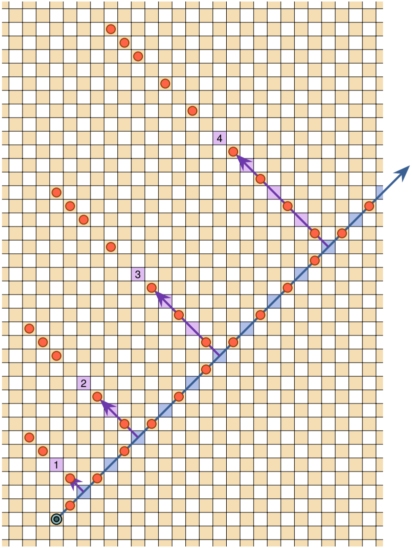


Black is obligated to jump.

Can jump infinitely,
but that is an immediate loss.

So will come
to rest at some square n.

Game value ω



Black is obligated to jump.

Can jump infinitely,
but that is an immediate loss.

So will come
to rest at some square n.

And then lose in n moves.

This is game value ω.

Infinite draughts exhibits high game values

Theorem (Hamkins & Leonessi [HL22; Leo21])

Every countable ordinal arises as the game value of a position in infinite draughts.

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Theorem (Hamkins & Leonessi [HL22; Leo21])

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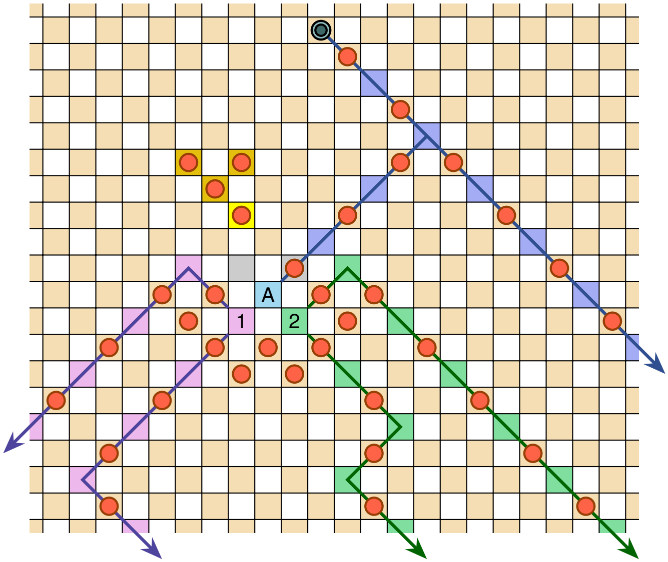
Main proof idea: embed certain well-founded trees into positions of infinite draughts.

The draughts play will unfold in a manner as though Black is climbing the tree, and consequently the game value will track the ordinal rank of the well-founded tree itself.


A
1
2

Infinite chess

Let us turn now to infinite chess.

Infinite chess

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Infinite chess is a game of the mind—we do not sit down in a café and play infinite chess over espresso.

Infinite chess

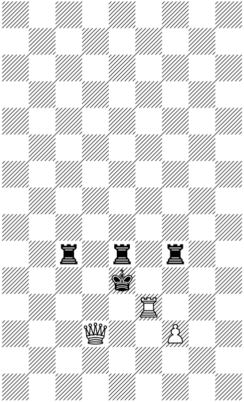
Let us turn now to infinite chess.

Infinite chess is a game of the mind—we do not sit down in a café and play infinite chess over espresso.

But rather, we sit down in that café and wonder what it would be like to play from this position or that one.

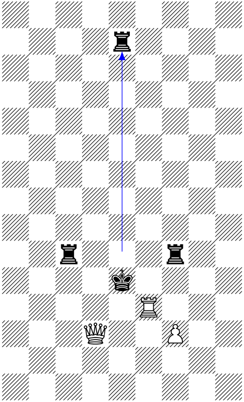
[EH14], [EHP17], [BHS12]

A finite position with value ω



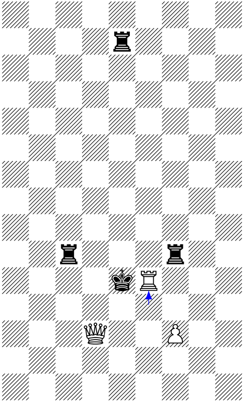
Black to move.

A finite position with value ω



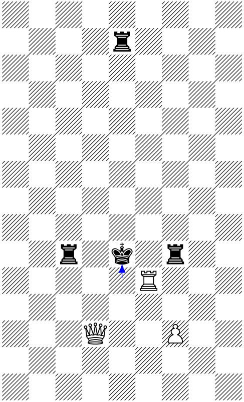
Black moves up arbitrary height

A finite position with value ω

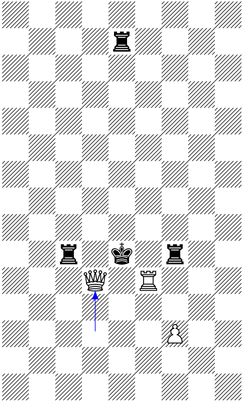


Check

A finite position with value ω



A finite position with value ω



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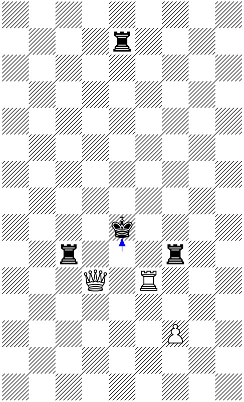
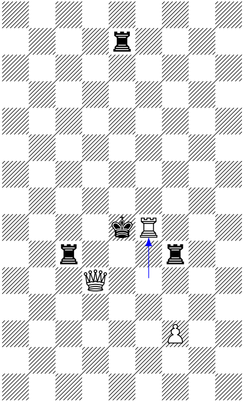


A finite position with value ω



Check

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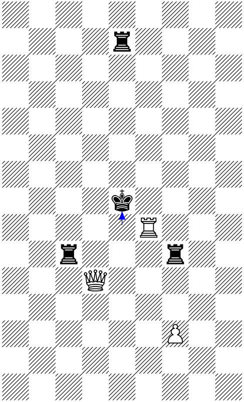
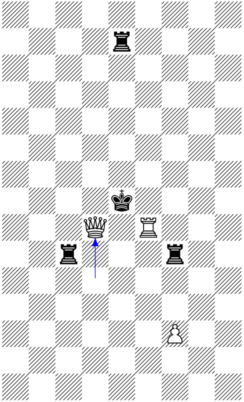


A finite position with value ω



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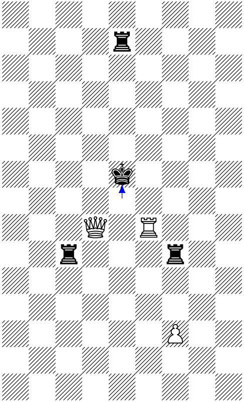
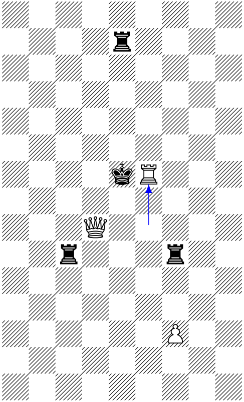
A finite position with value ω

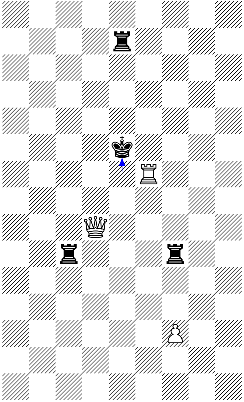


A finite position with value ω

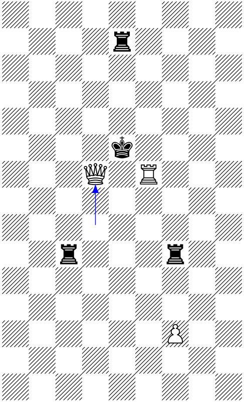


Check

A finite position with value ω

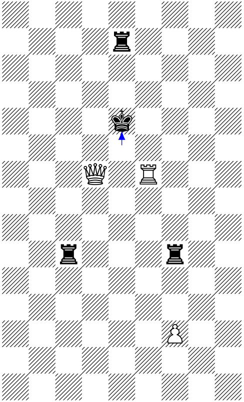


A finite position with value ω

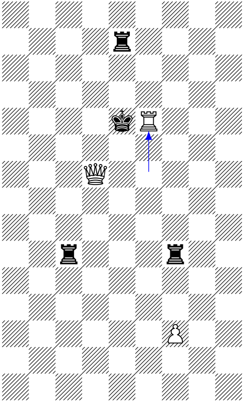


Check

A finite position with value ω



A finite position with value ω



Check

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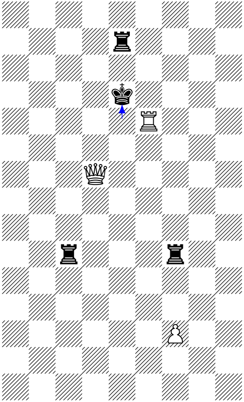
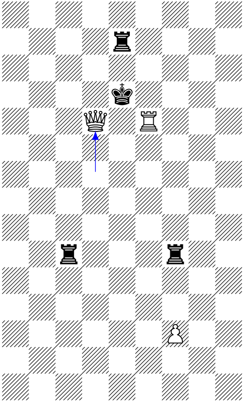


A finite position with value ω



Check

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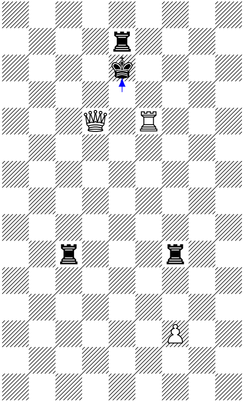

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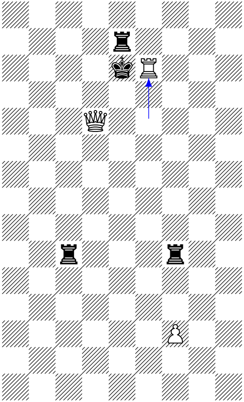
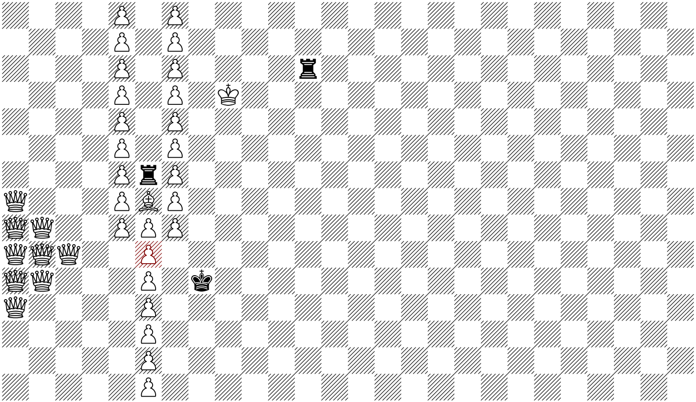


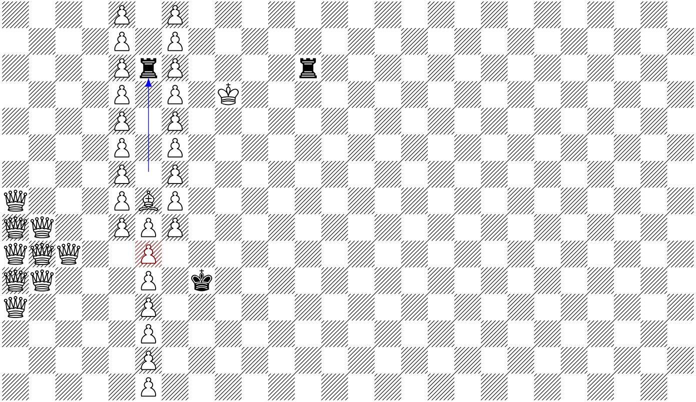
Checkmate. Black can cause arbitrary delay, but the only choice is on first move, so the initial value is ω.

Releasing the Hordes, with value ω^2



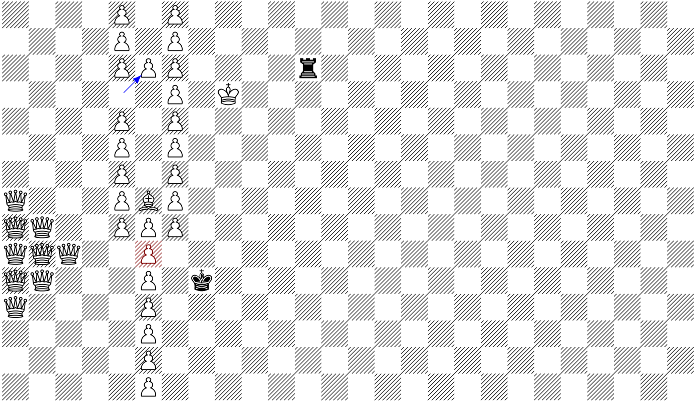
Black to move.

Releasing the Hordes, with value ω^2



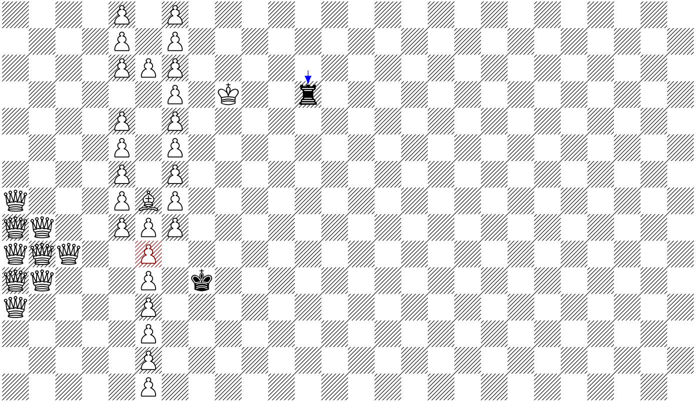
He moves trapped rook up arbitrary height.

Releasing the Hordes, with value ω^2



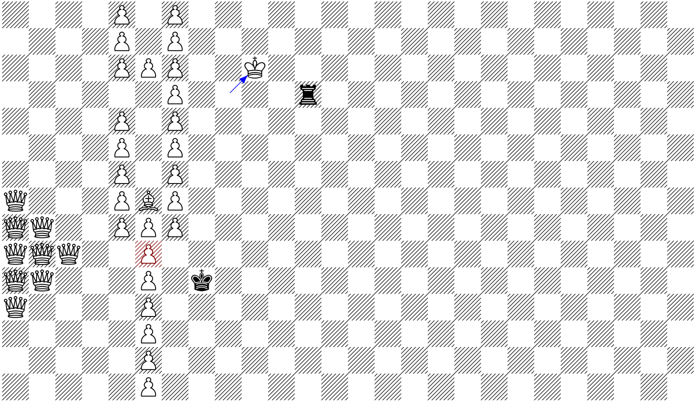
White should capture from left side.

Releasing the Hordes, with value ω^2



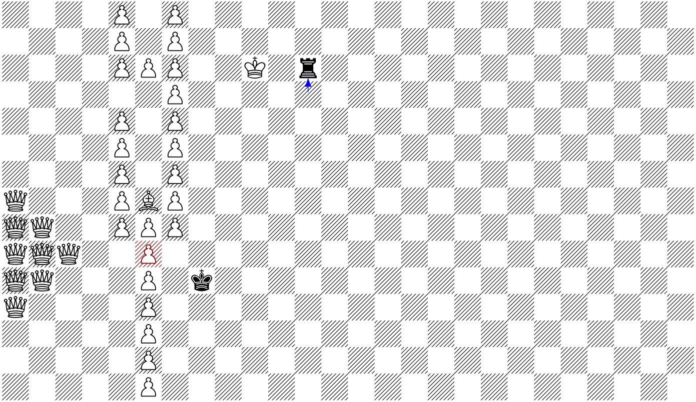
Now black begins to harass white king.

Releasing the Hordes, with value ω^2



White must chase down the rook to avoid perpetual check.

Releasing the Hordes, with value ω^2



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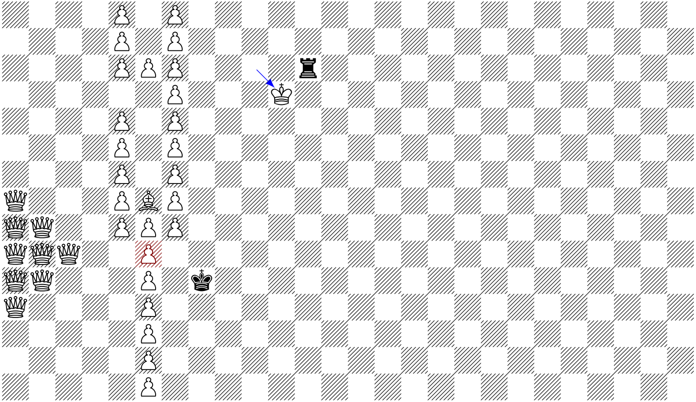

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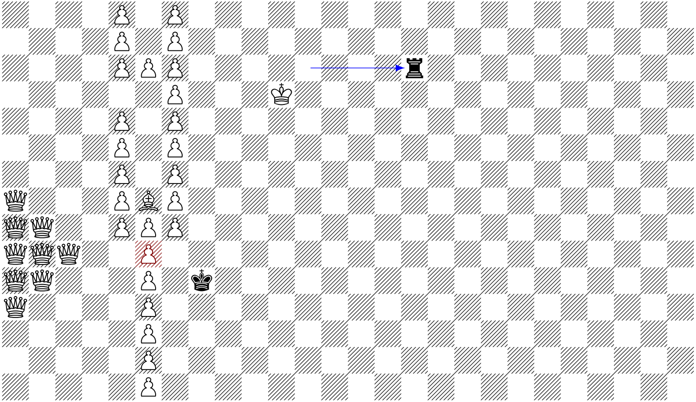
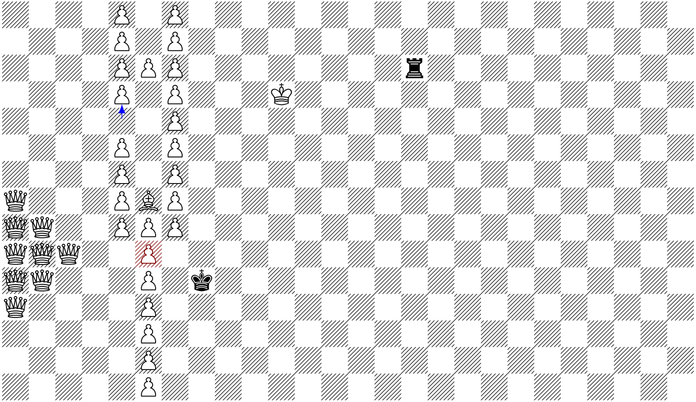


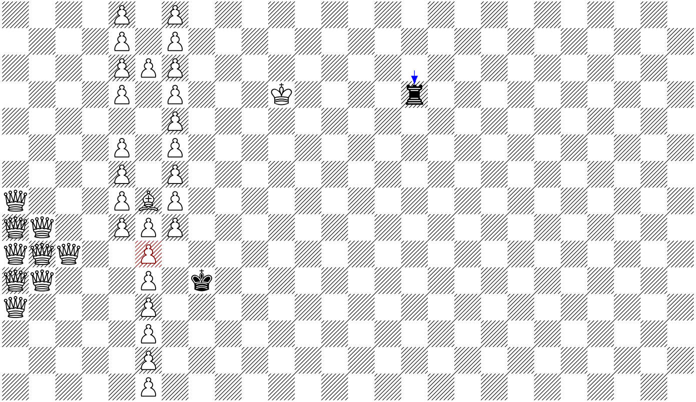
Black must move away to save rook.

Releasing the Hordes, with value ω^2



Now is white's chance to advance a pawn.

Releasing the Hordes, with value ω^2



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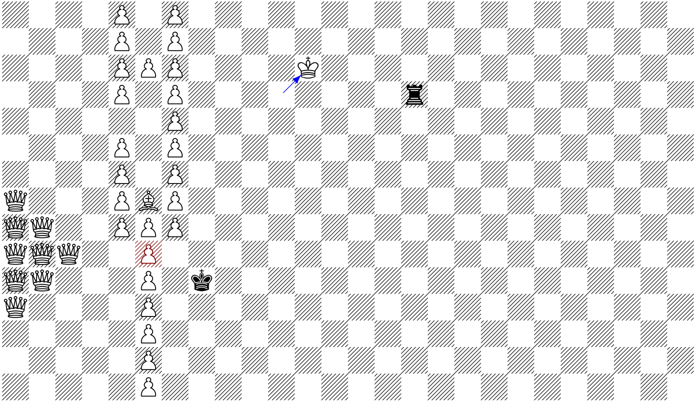

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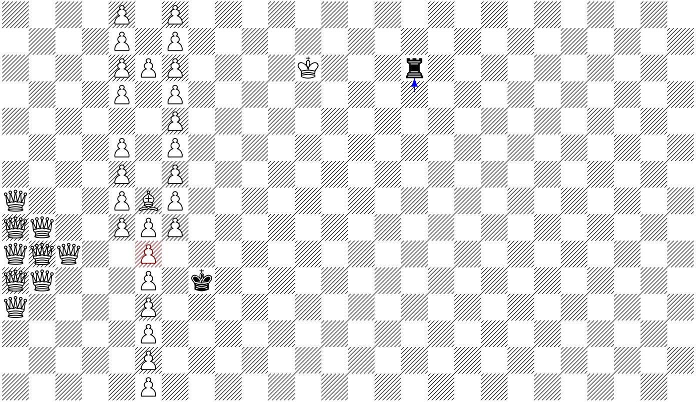

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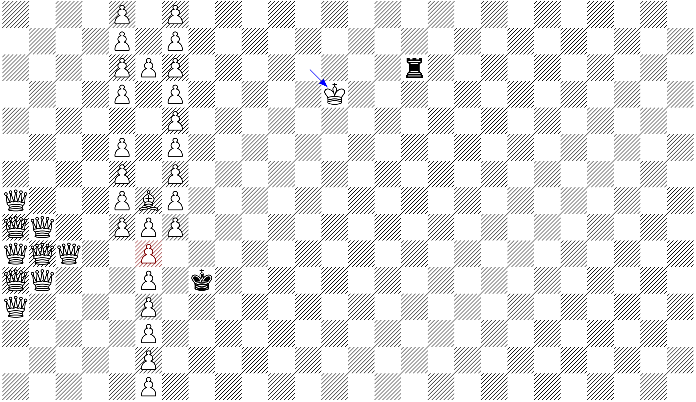

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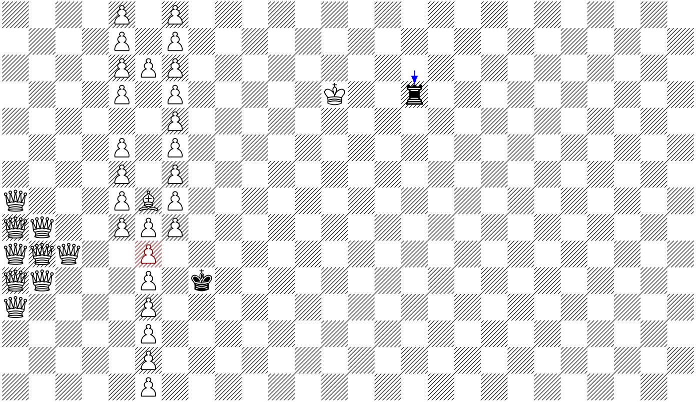

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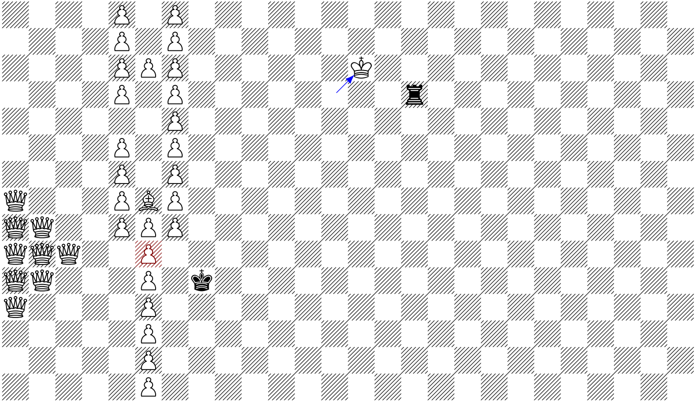

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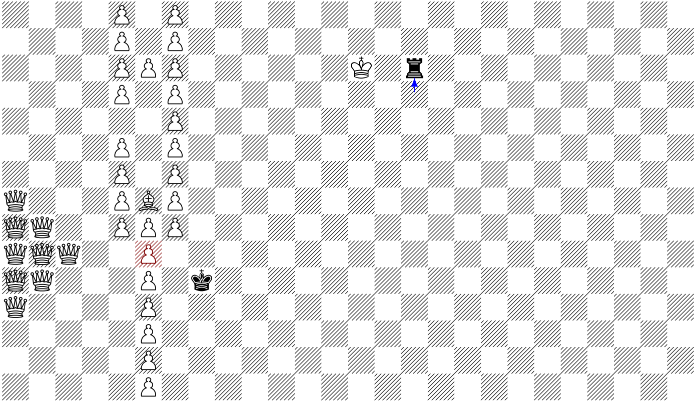

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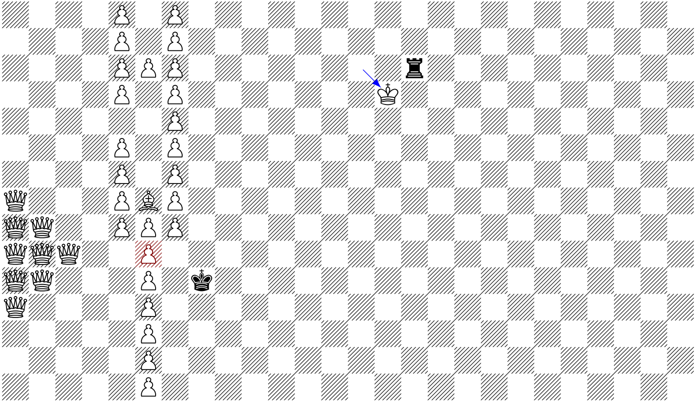

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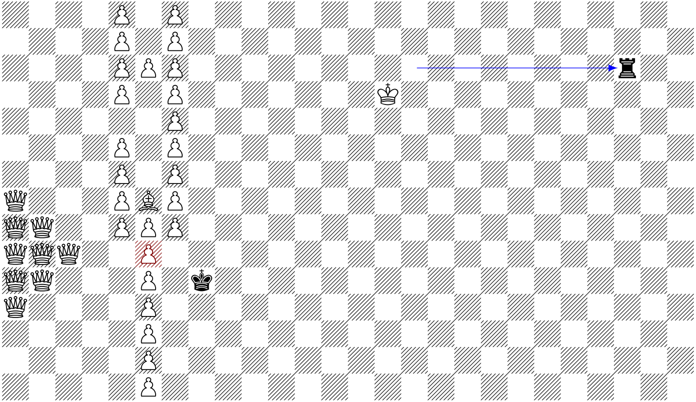
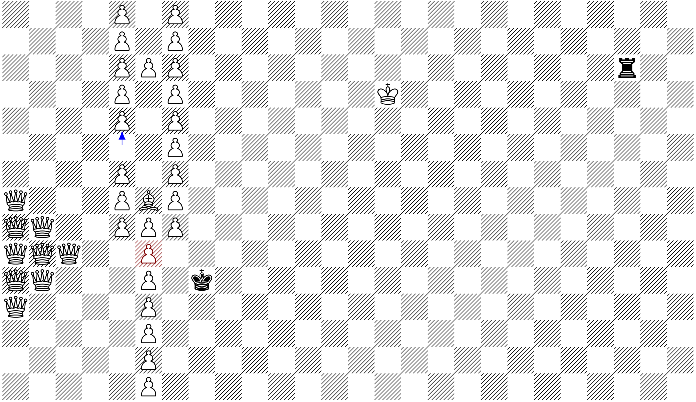


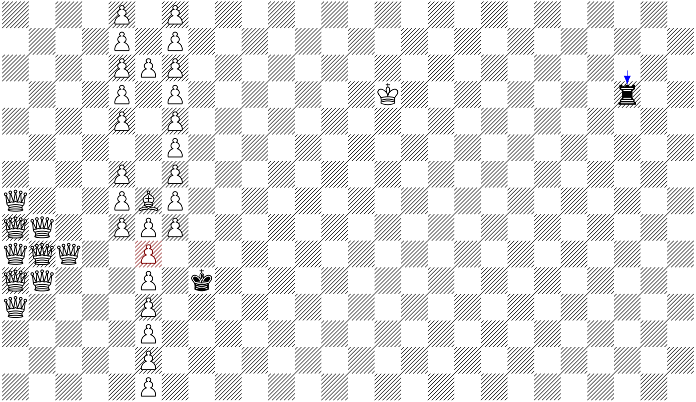
Black moves arbitrary distance out.

Releasing the Hordes, with value ω^2



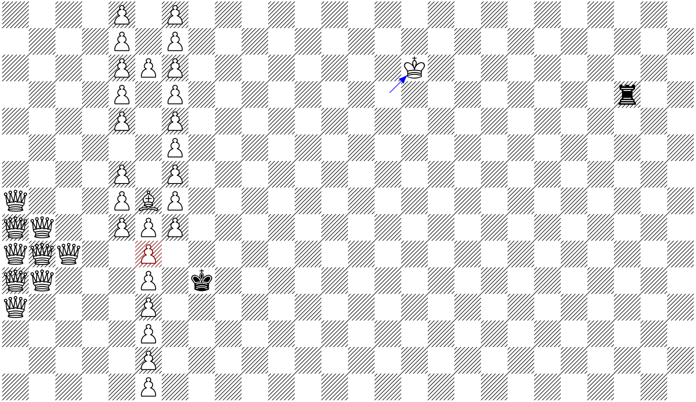
Another chance to advance a pawn.

Releasing the Hordes, with value ω^2



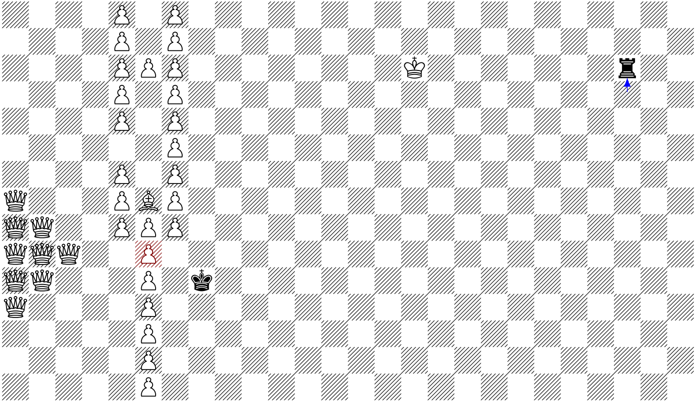
Black harasses the white king.

Releasing the Hordes, with value ω^2



White must chase him down.

Releasing the Hordes, with value ω^2



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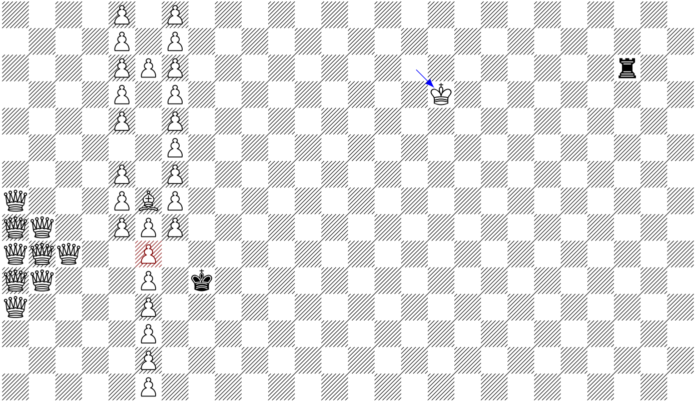

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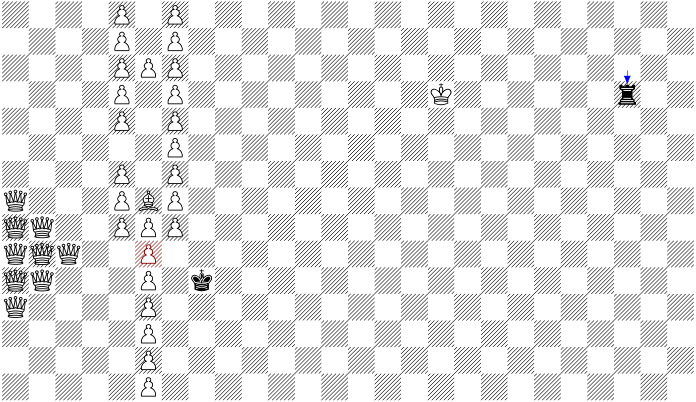

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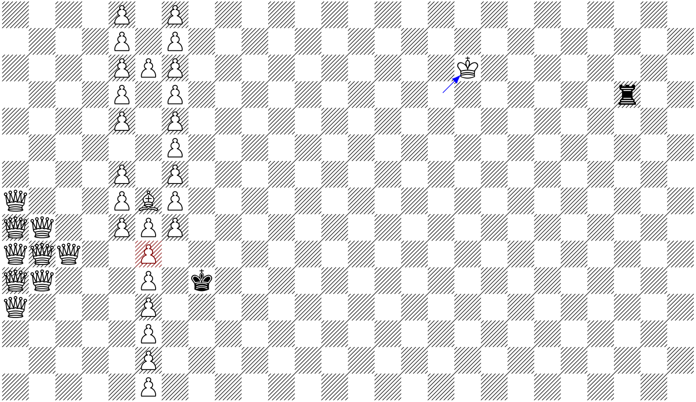

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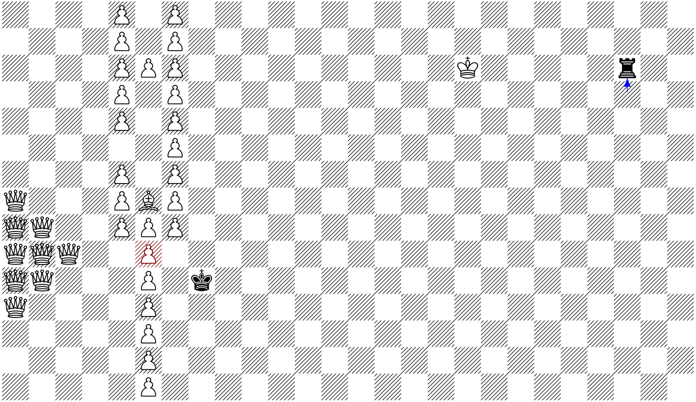

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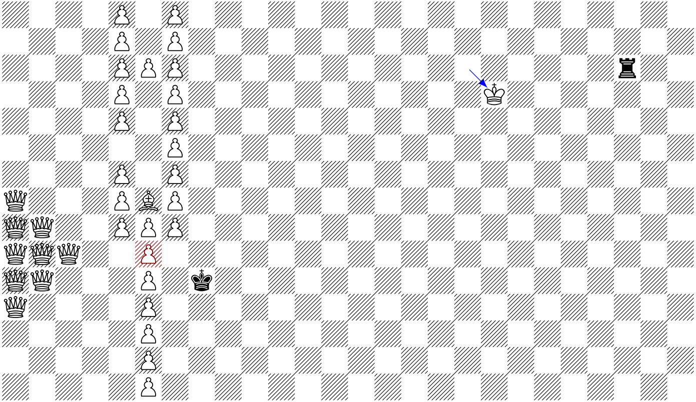

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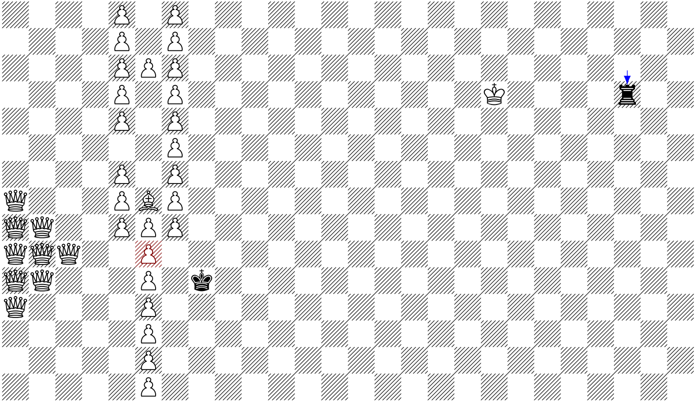

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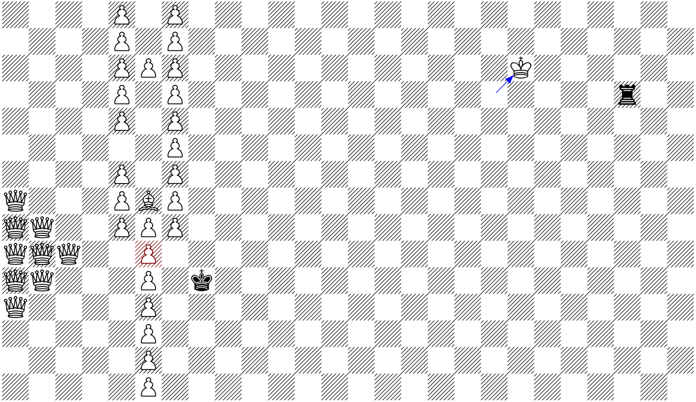

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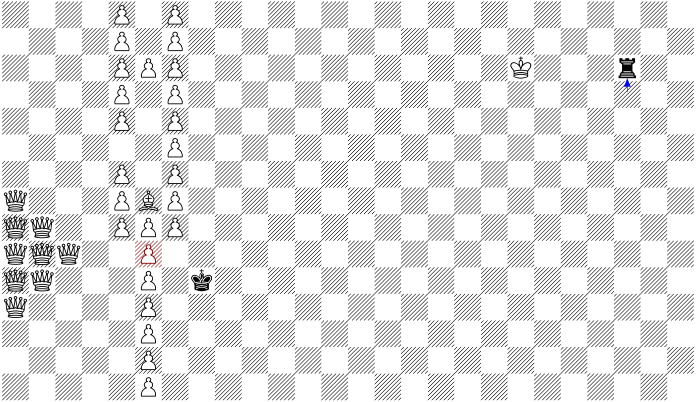

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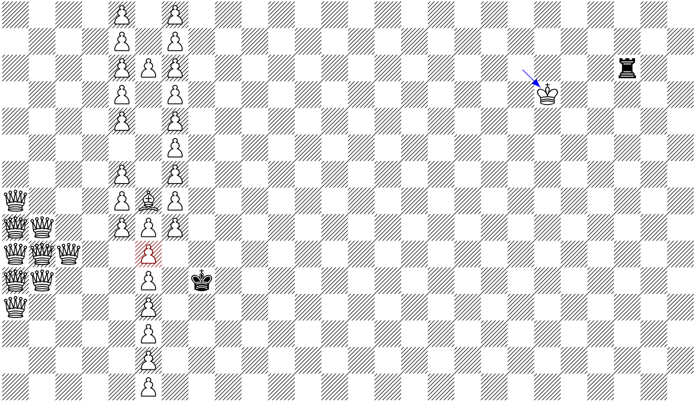

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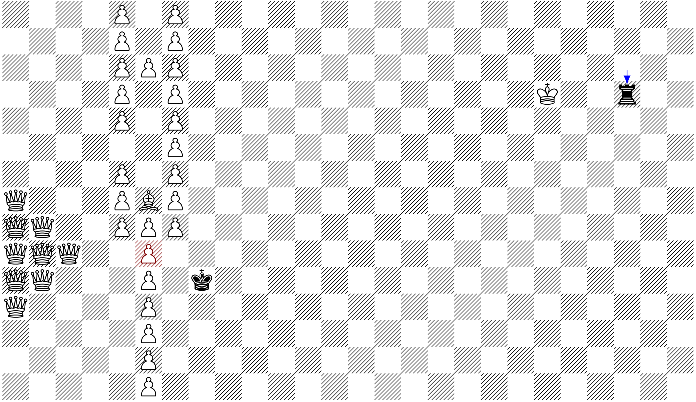

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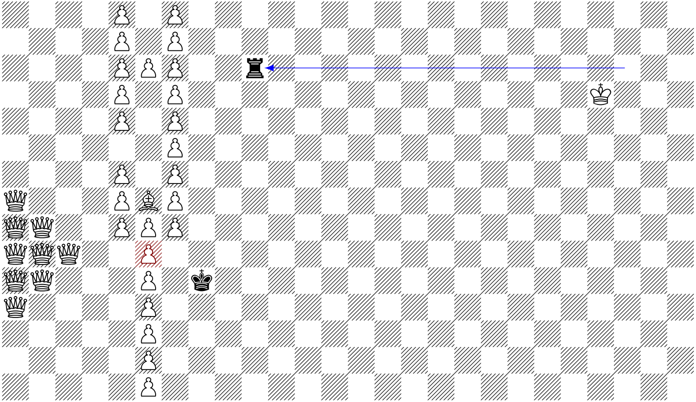
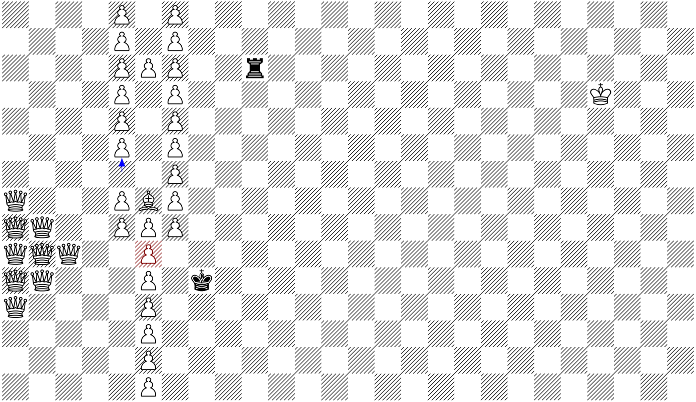


(Black should actually move arbitrary distance to the right.)

Releasing the Hordes, with value ω^2



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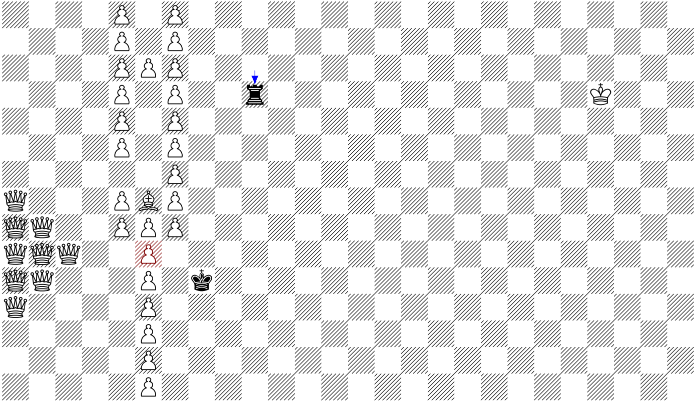

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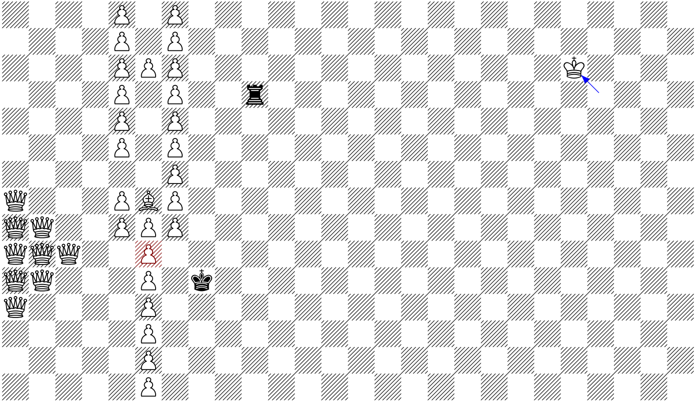

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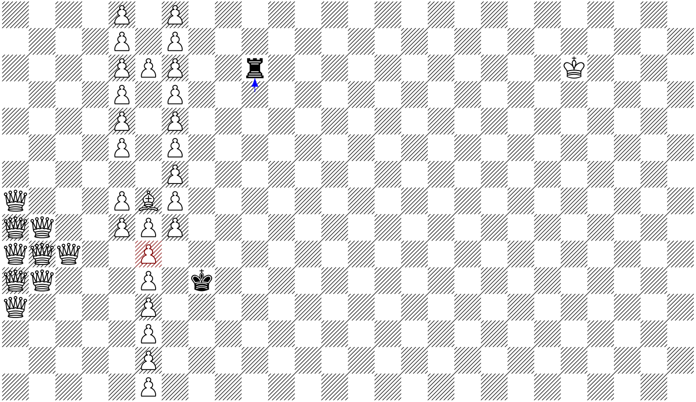

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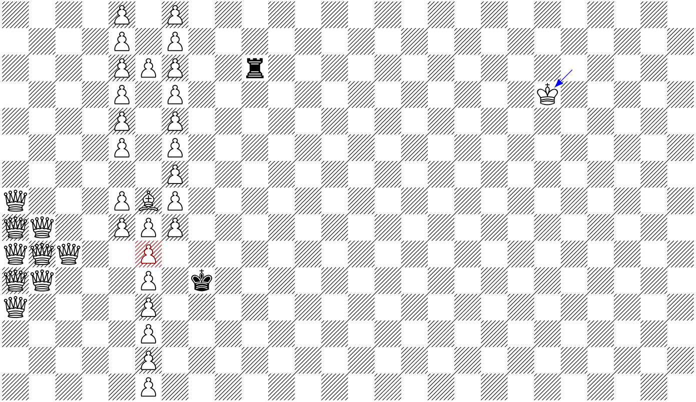

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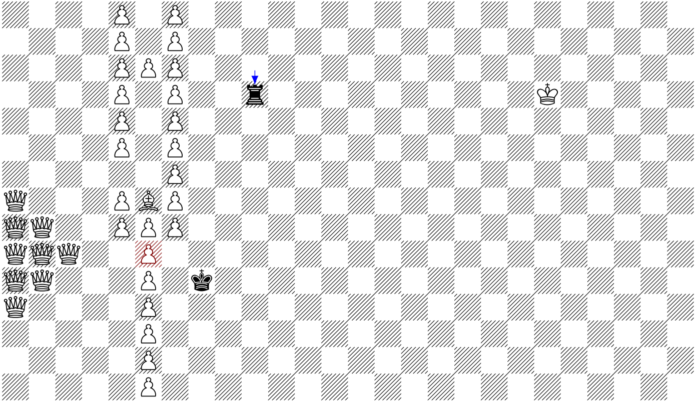

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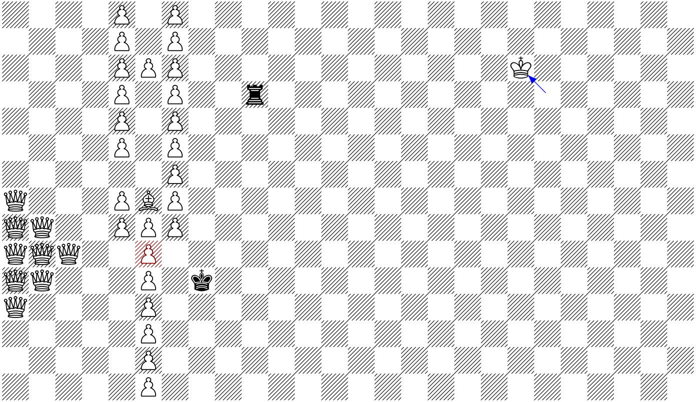

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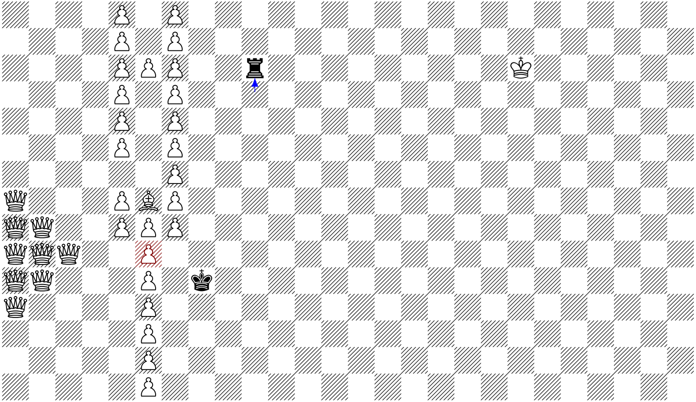

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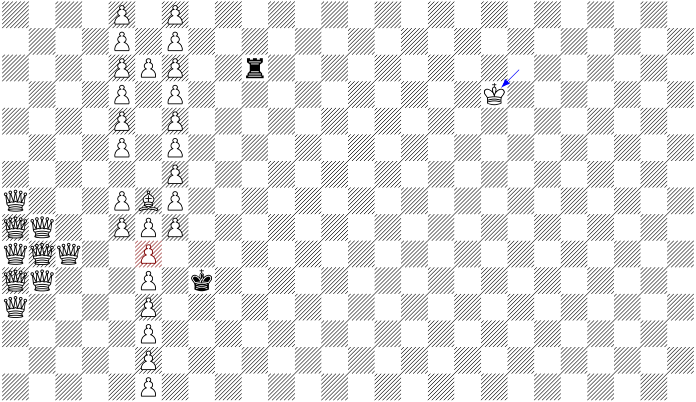
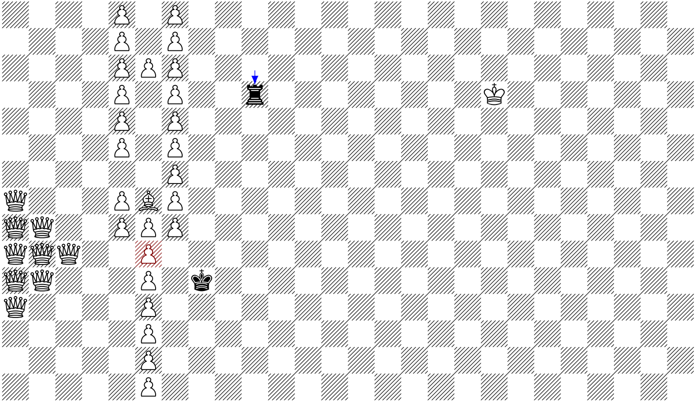
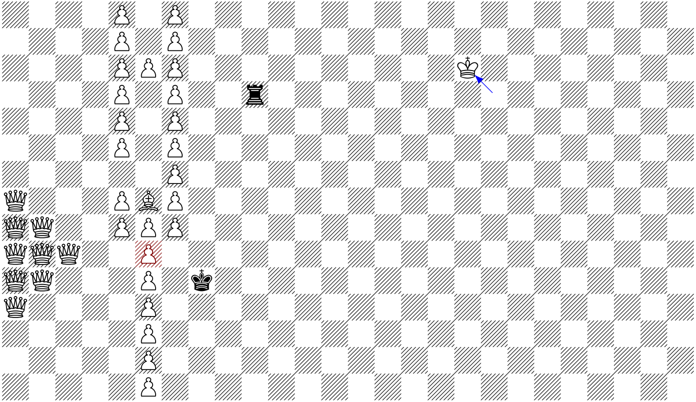


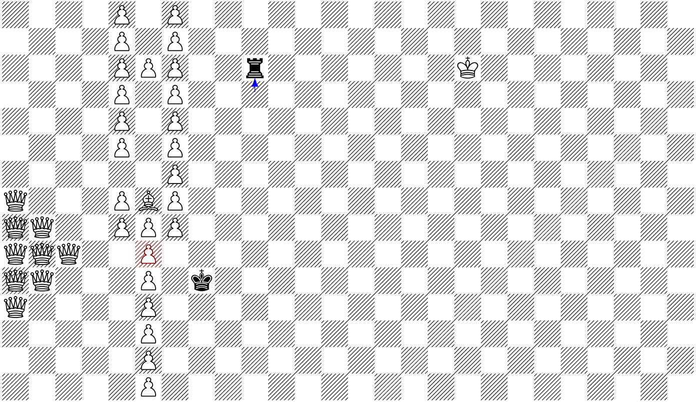
Releasing the Hordes, with value ω^2



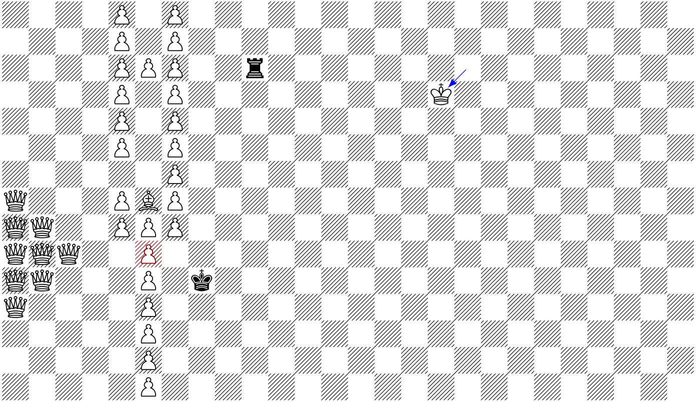
Releasing the Hordes, with value ω^2



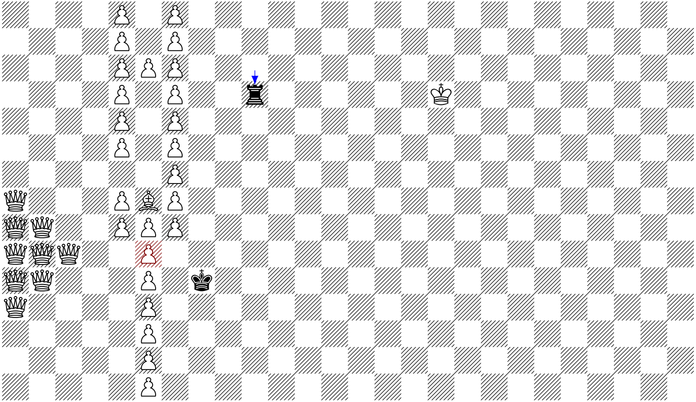
Releasing the Hordes, with value ω^2



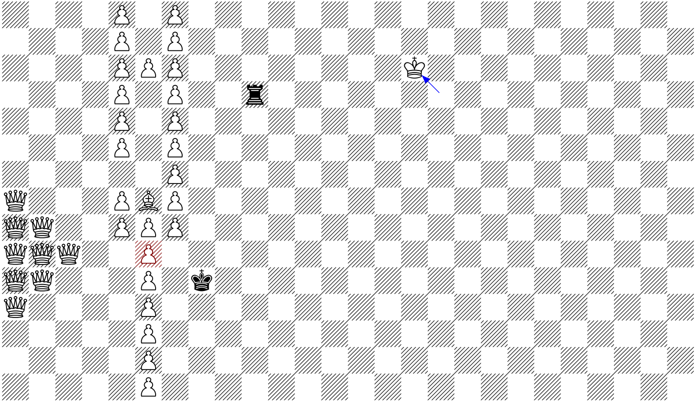
Releasing the Hordes, with value ω^2



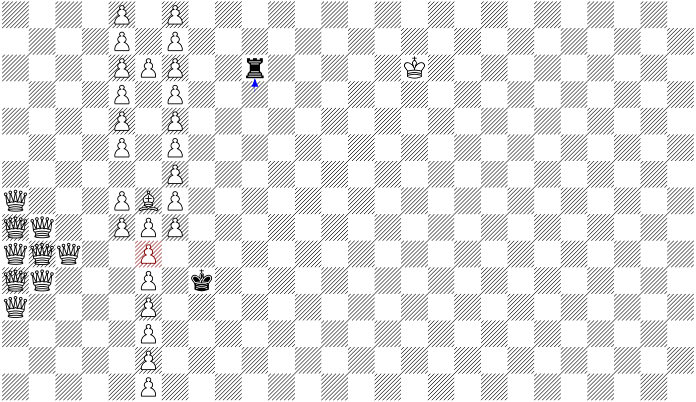
Releasing the Hordes, with value ω^2



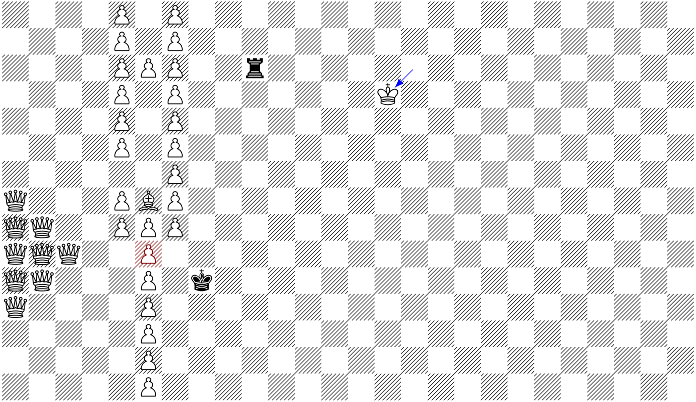
Releasing the Hordes, with value ω^2



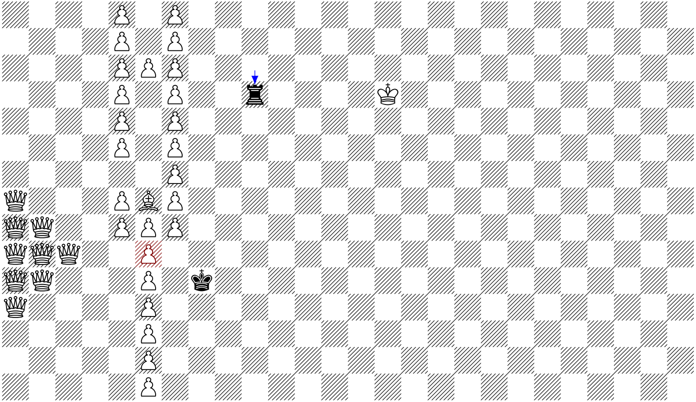
Releasing the Hordes, with value ω^2



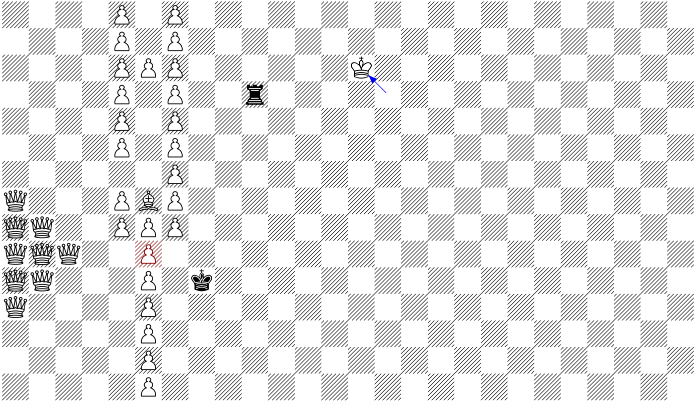
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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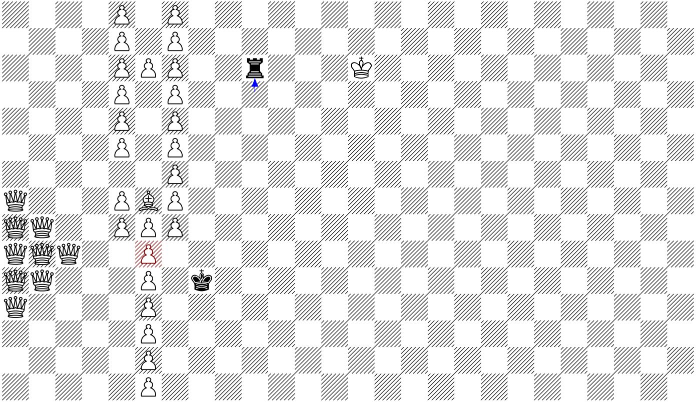
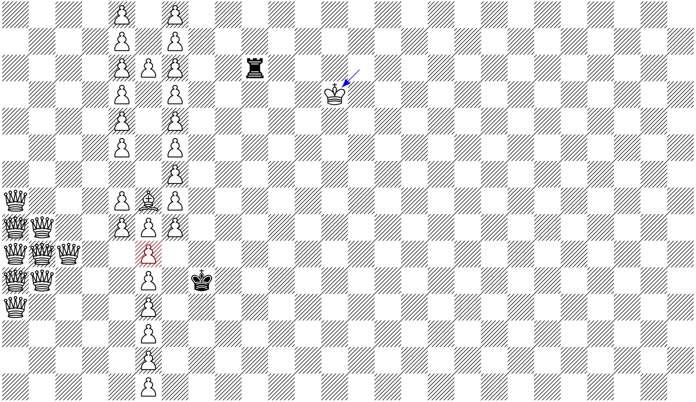
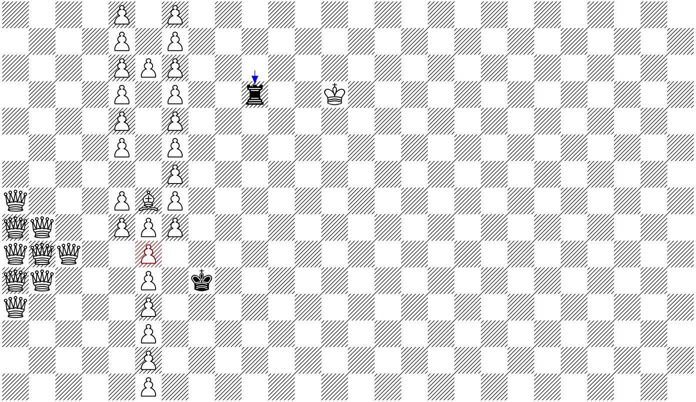


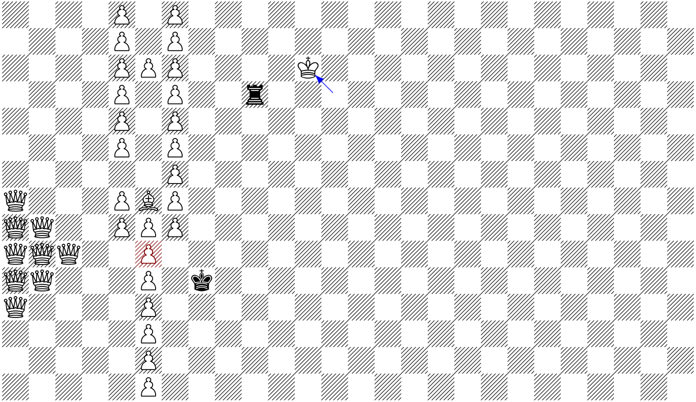
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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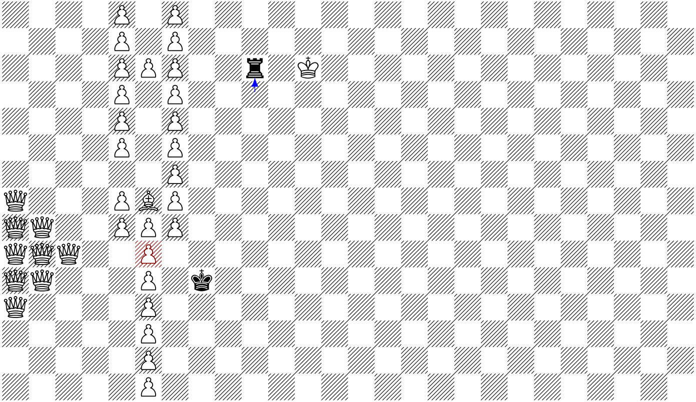
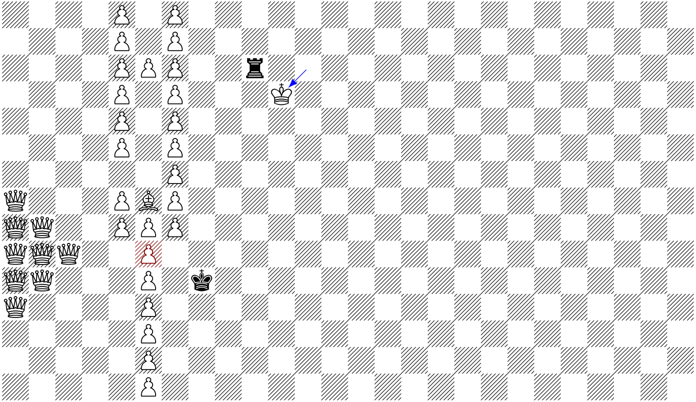
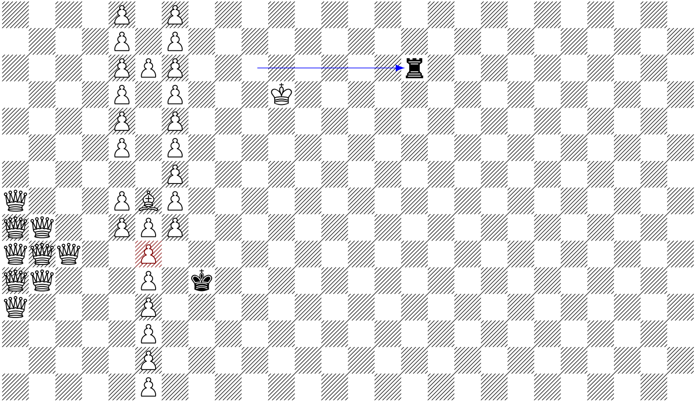


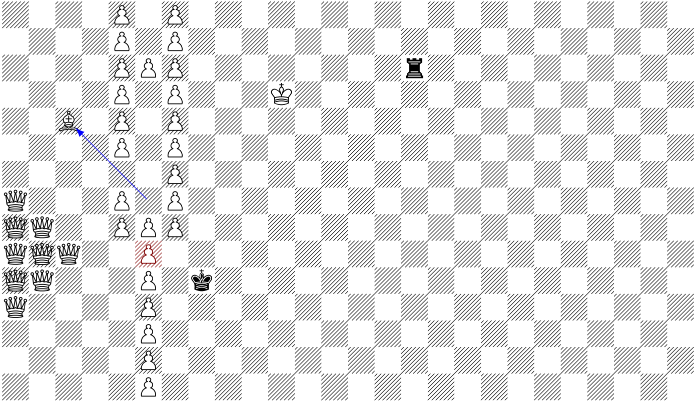
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

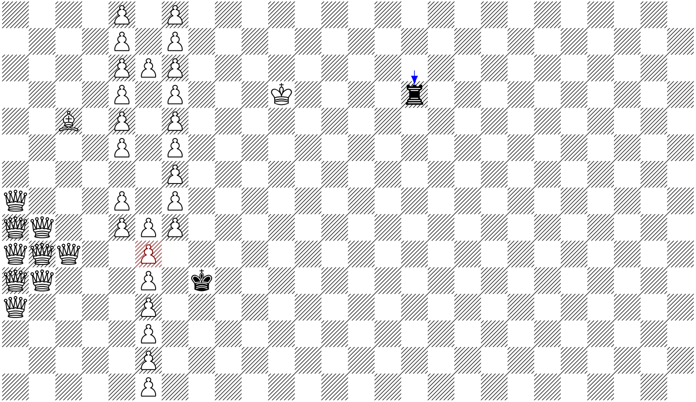


Releasing the Hordes, with value ω^2

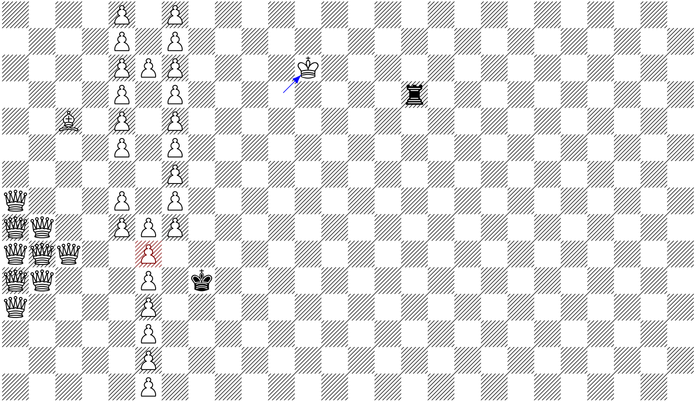


The bishop unlocks the door.

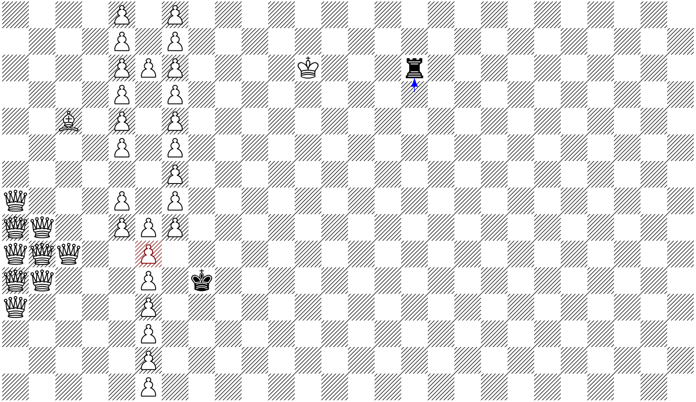
Releasing the Hordes, with value ω^2



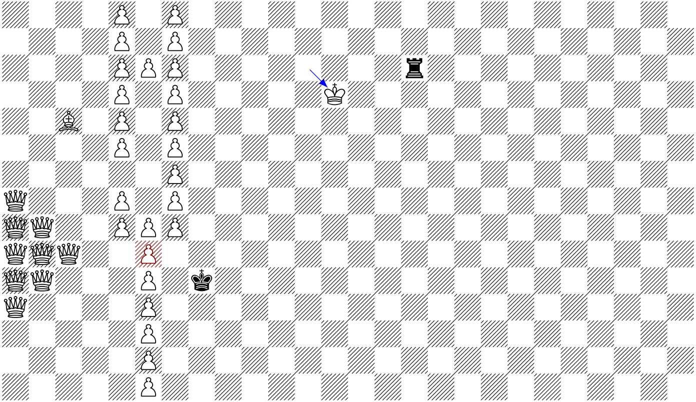
Releasing the Hordes, with value ω^2



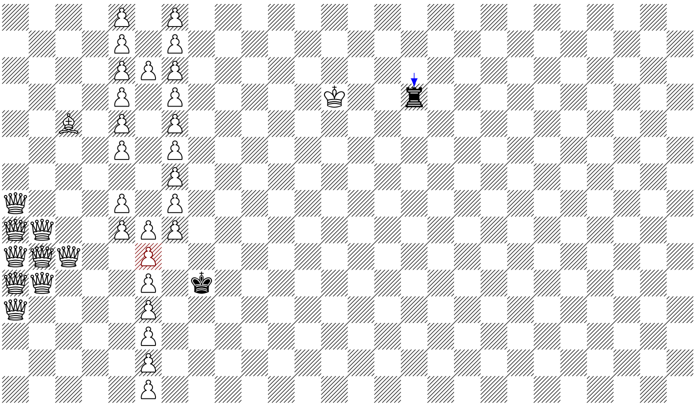
Releasing the Hordes, with value ω^2



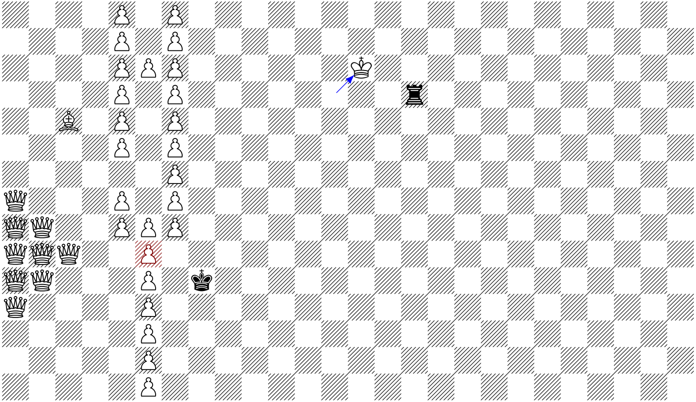
Releasing the Hordes, with value ω^2



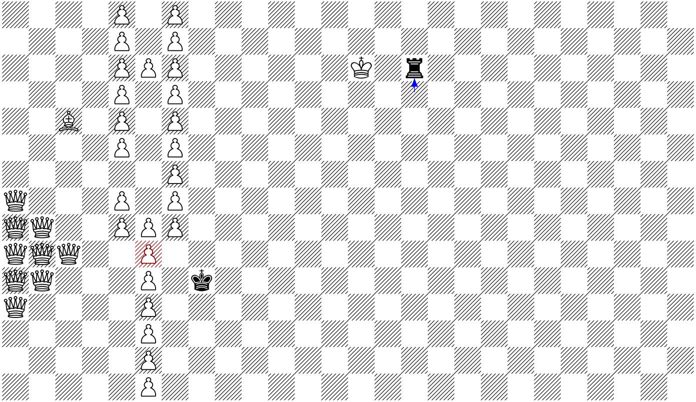
Releasing the Hordes, with value ω^2



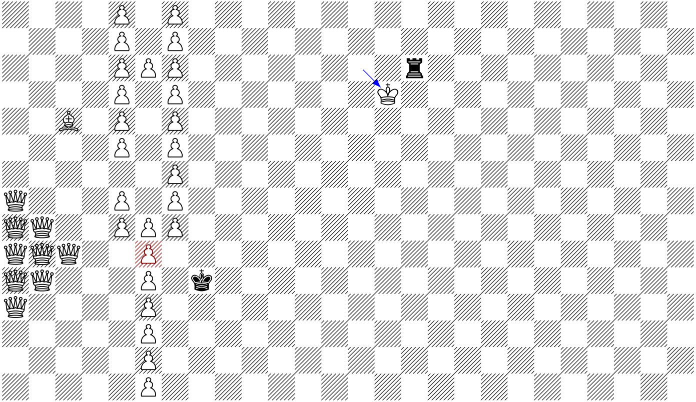
Releasing the Hordes, with value ω^2



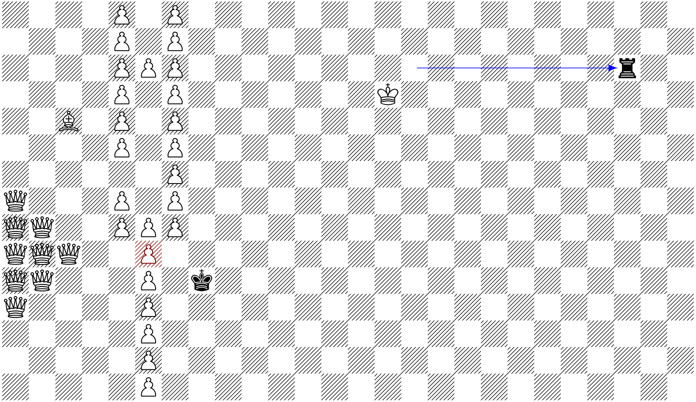
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

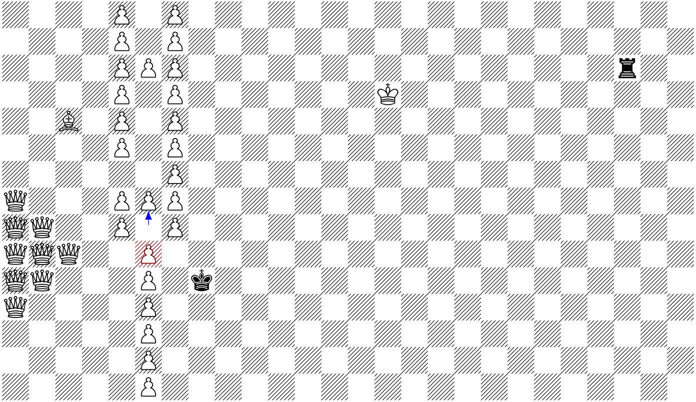


Releasing the Hordes, with value ω^2

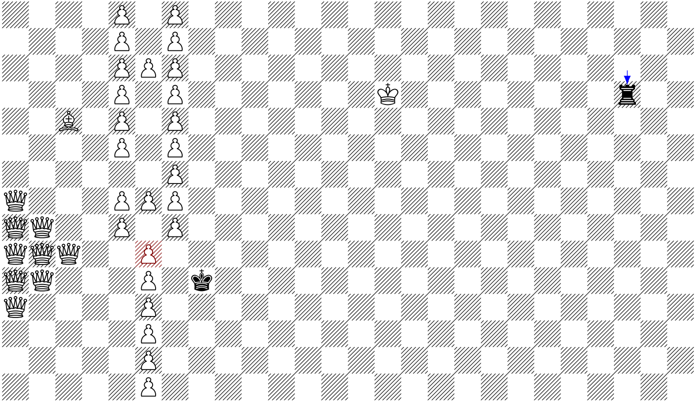


Black can move rook arbitrary distance.

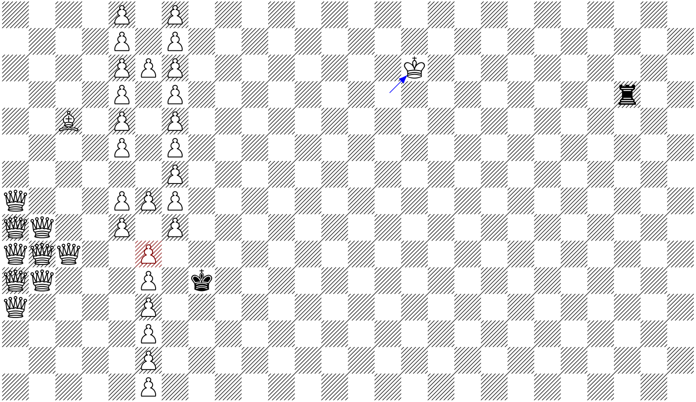
Releasing the Hordes, with value ω^2



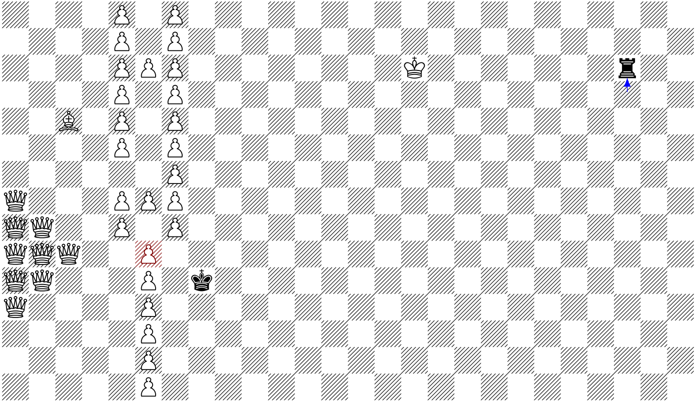
Releasing the Hordes, with value ω^2



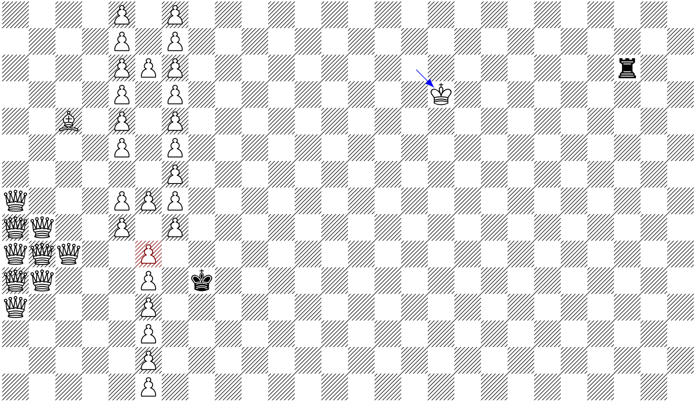
Releasing the Hordes, with value ω^2



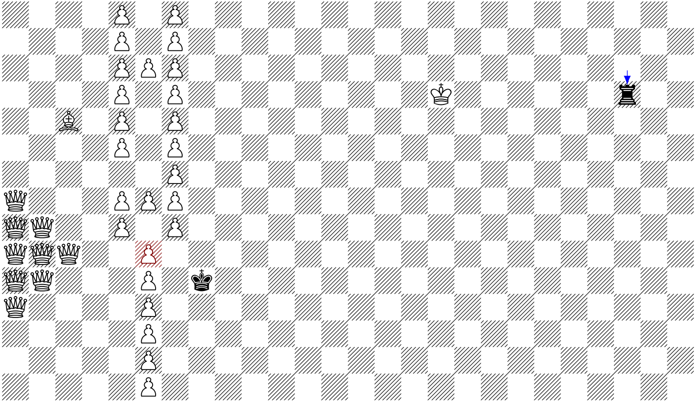
Releasing the Hordes, with value ω^2



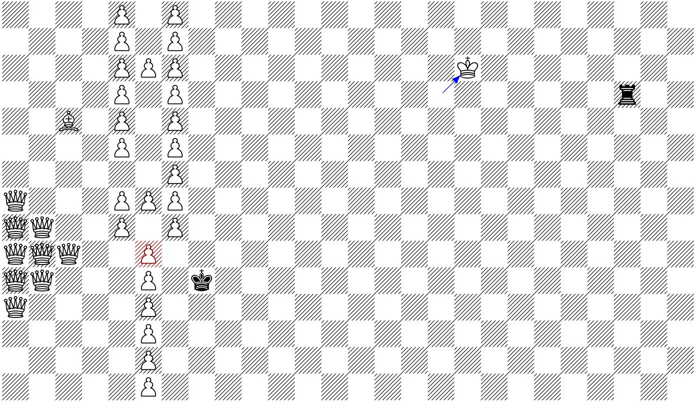
Releasing the Hordes, with value ω^2



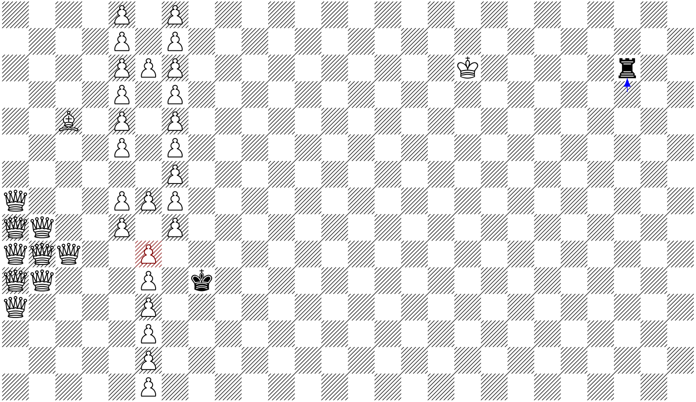
Releasing the Hordes, with value ω^2



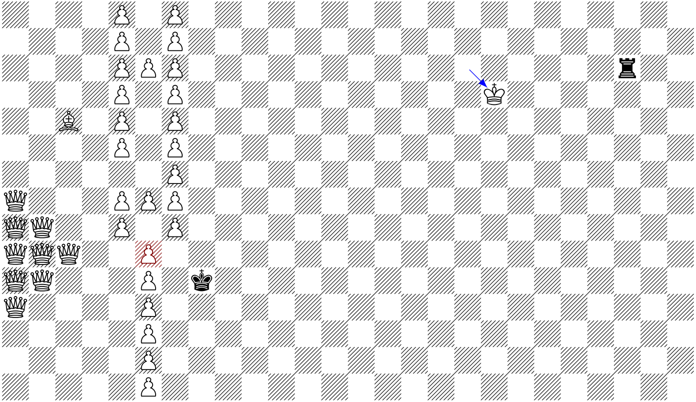
Releasing the Hordes, with value ω^2



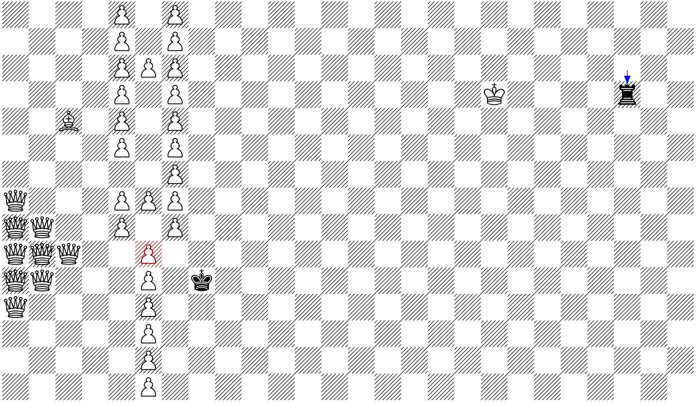
Releasing the Hordes, with value ω^2



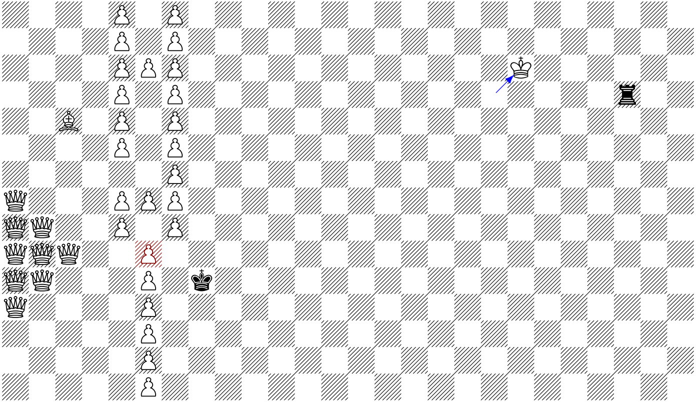
Releasing the Hordes, with value ω^2



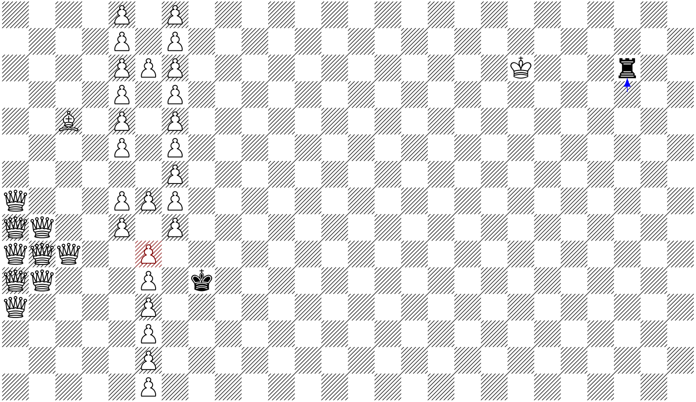
Releasing the Hordes, with value ω^2



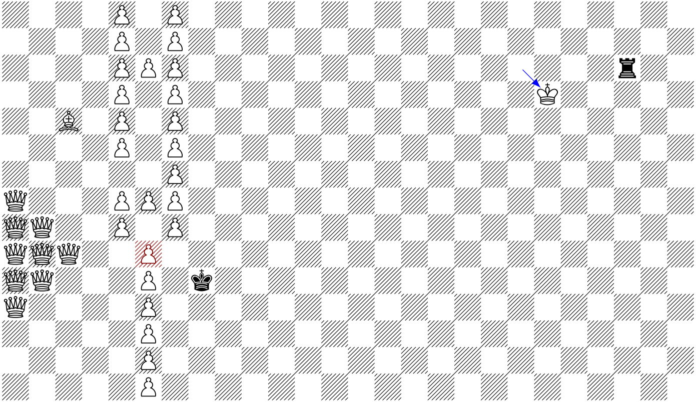
Releasing the Hordes, with value ω^2



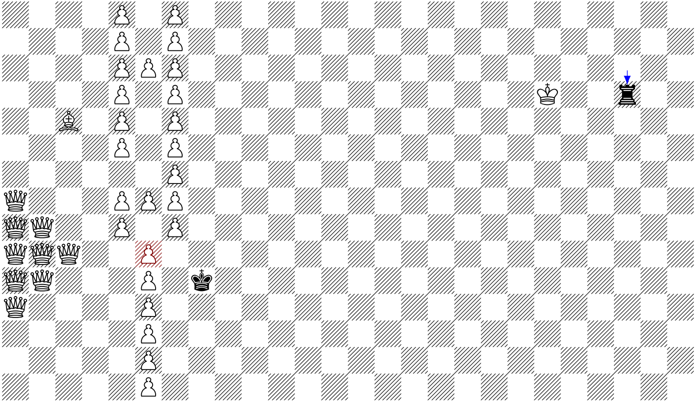
Releasing the Hordes, with value ω^2



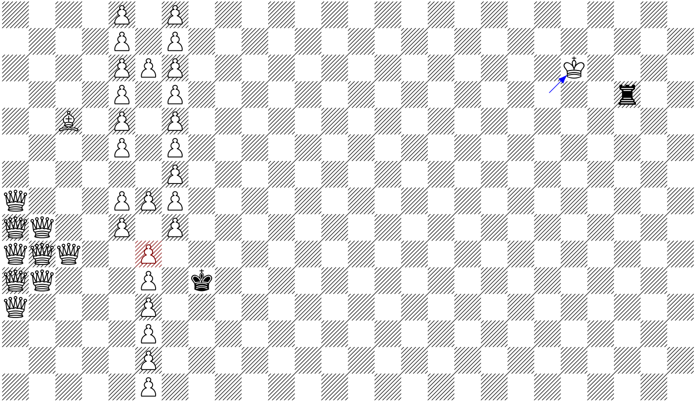
Releasing the Hordes, with value ω^2



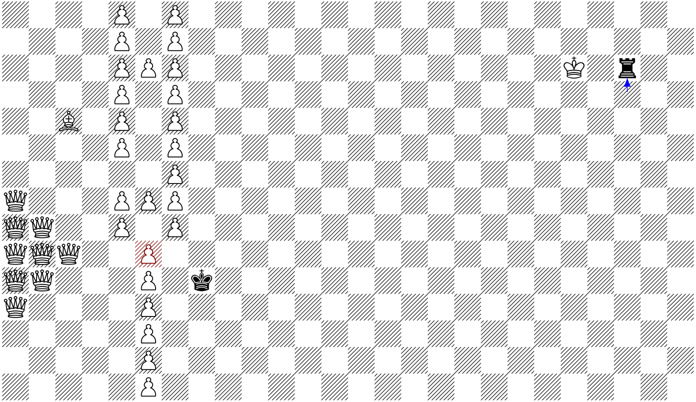
Releasing the Hordes, with value ω^2



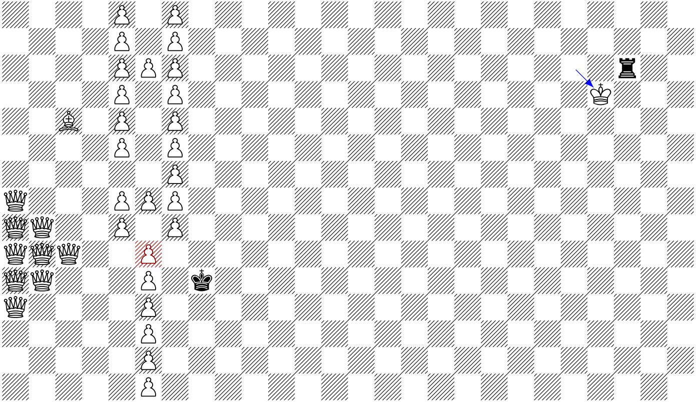
Releasing the Hordes, with value ω^2



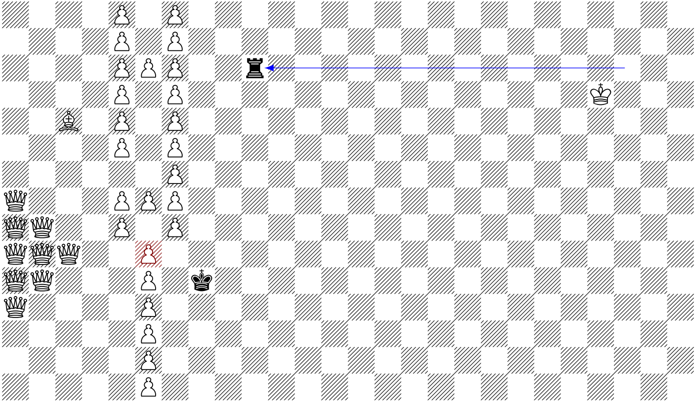
Releasing the Hordes, with value ω^2



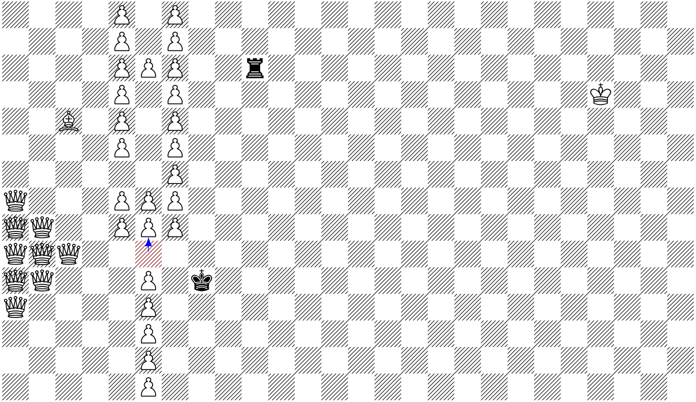
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

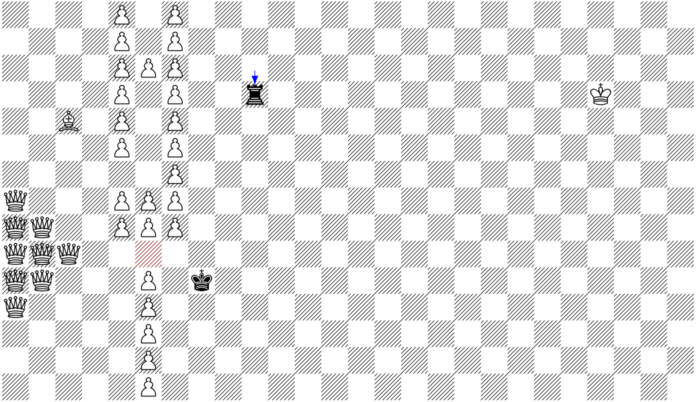


Releasing the Hordes, with value ω^2

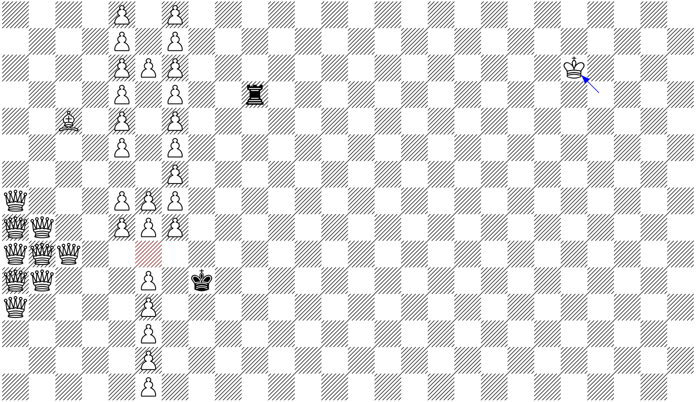


The portcullis opens...

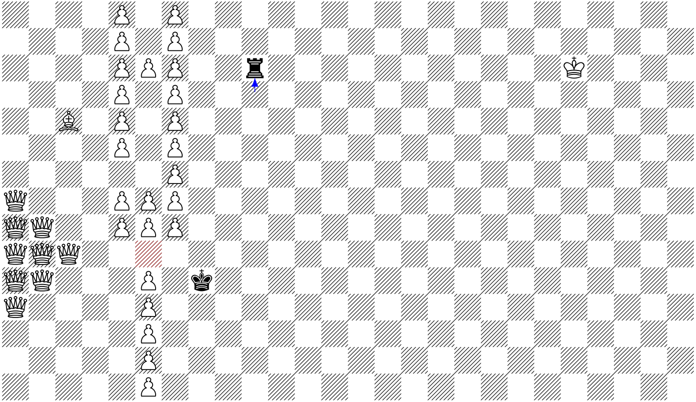
Releasing the Hordes, with value ω^2



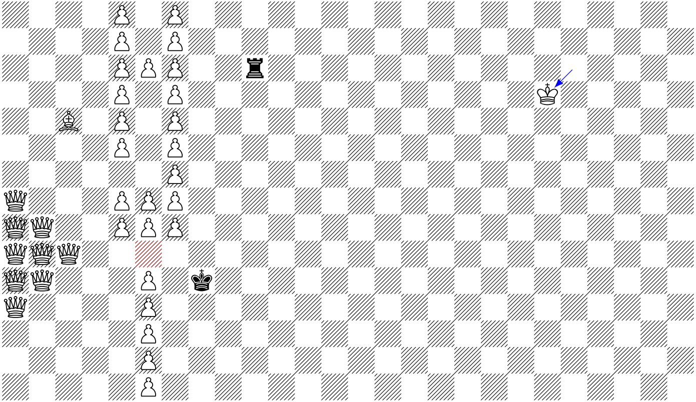
Releasing the Hordes, with value ω^2



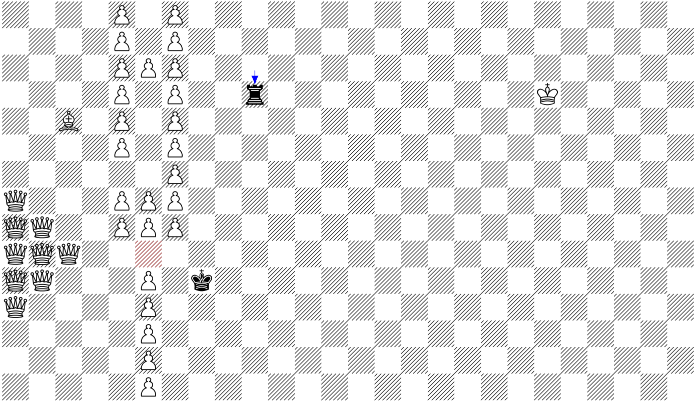
Releasing the Hordes, with value ω^2



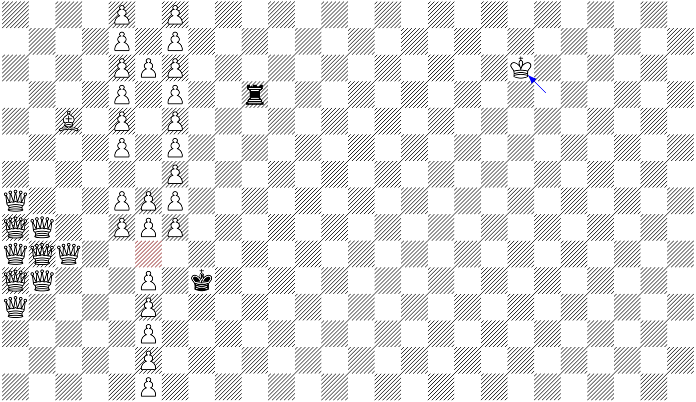
Releasing the Hordes, with value ω^2



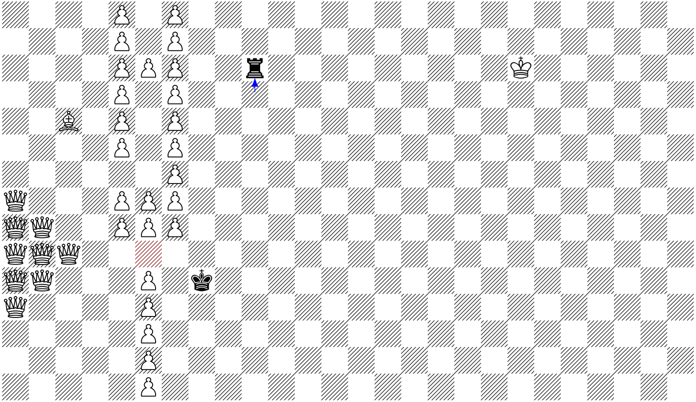
Releasing the Hordes, with value ω^2



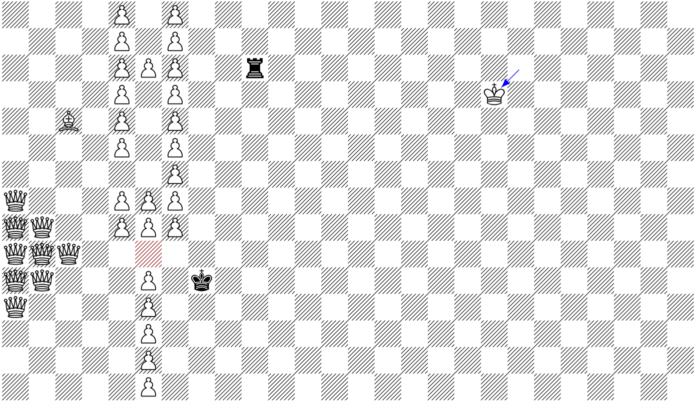
Releasing the Hordes, with value ω^2



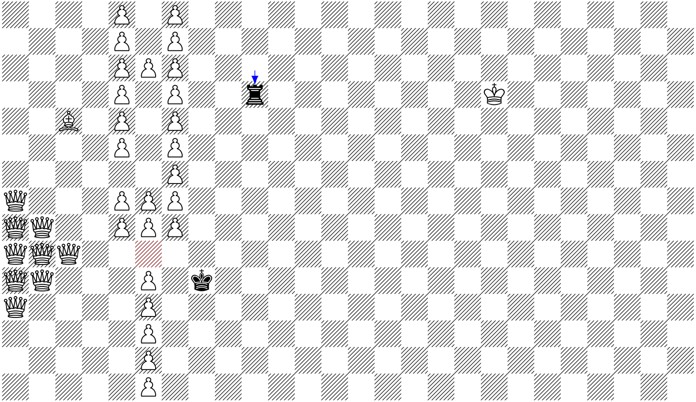
Releasing the Hordes, with value ω^2



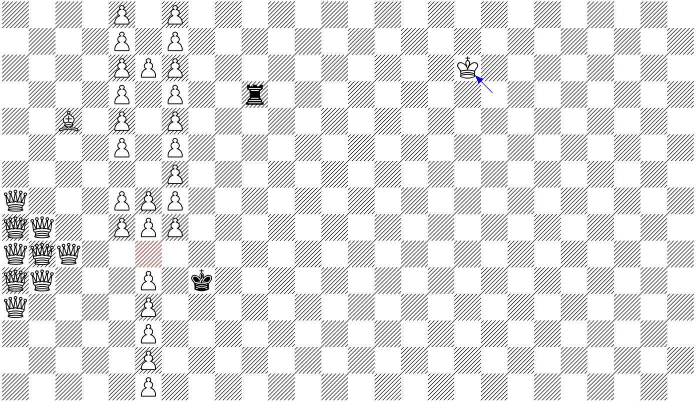
Releasing the Hordes, with value ω^2



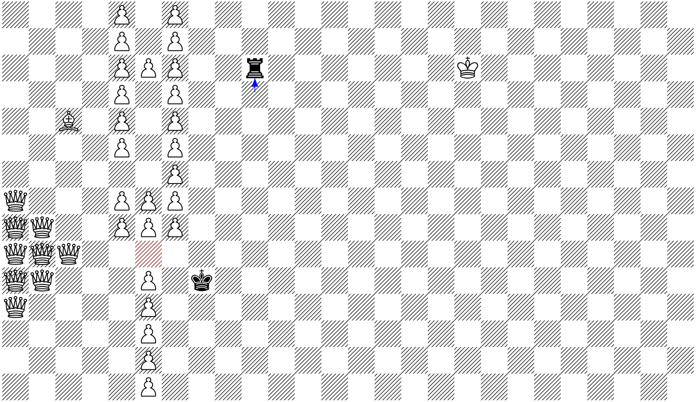
Releasing the Hordes, with value ω^2



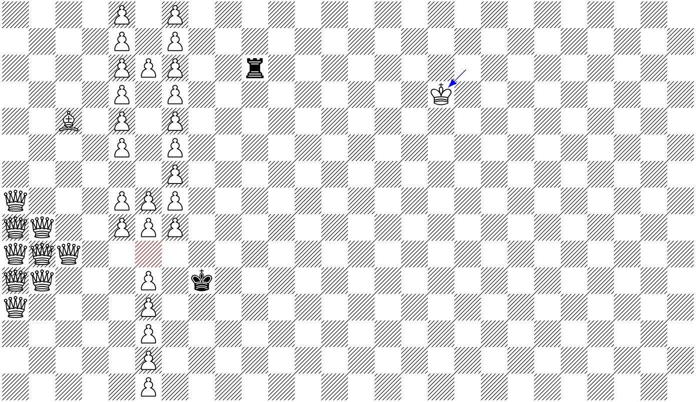
Releasing the Hordes, with value ω^2



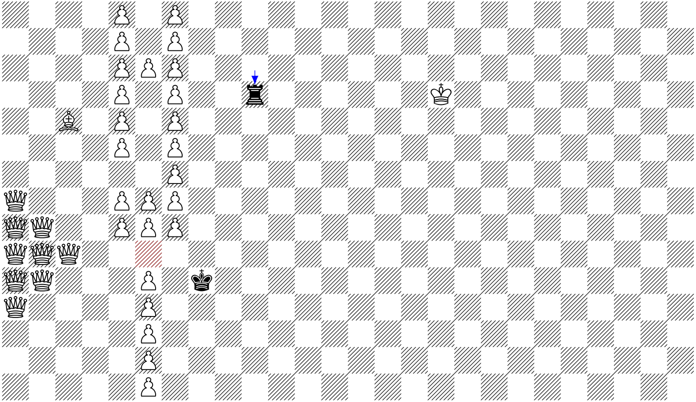
Releasing the Hordes, with value ω^2



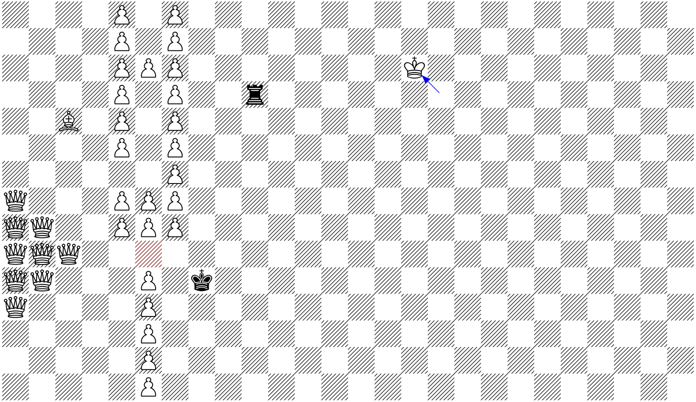
Releasing the Hordes, with value ω^2



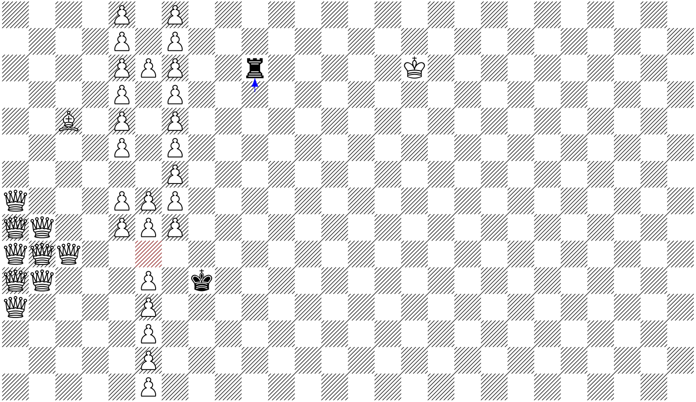
Releasing the Hordes, with value ω^2



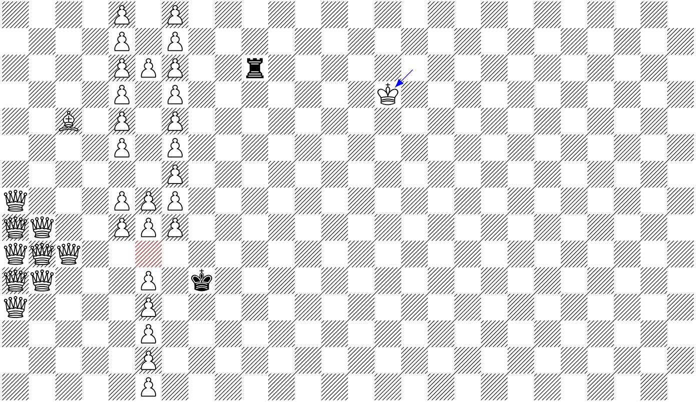
Releasing the Hordes, with value ω^2



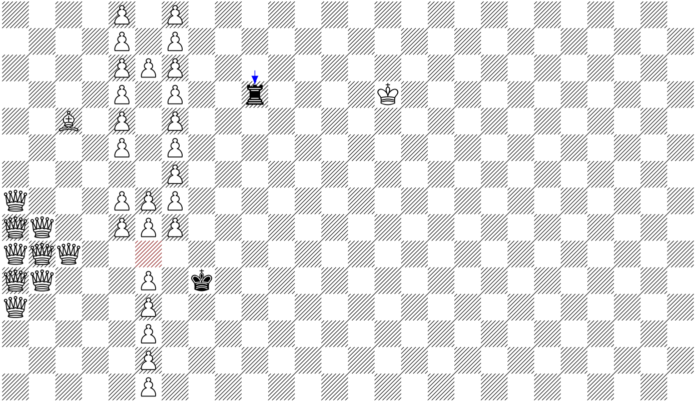
Releasing the Hordes, with value ω^2



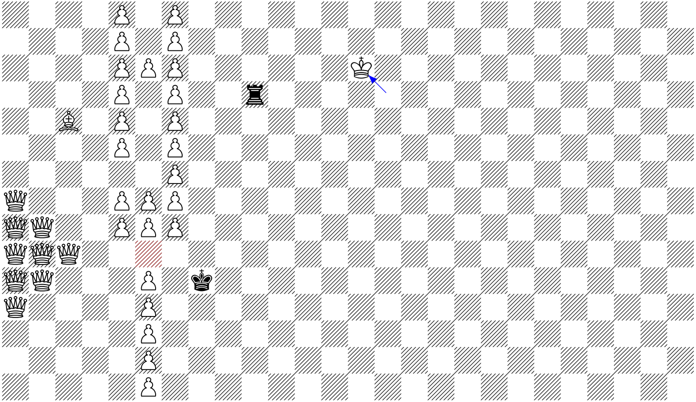
Releasing the Hordes, with value ω^2



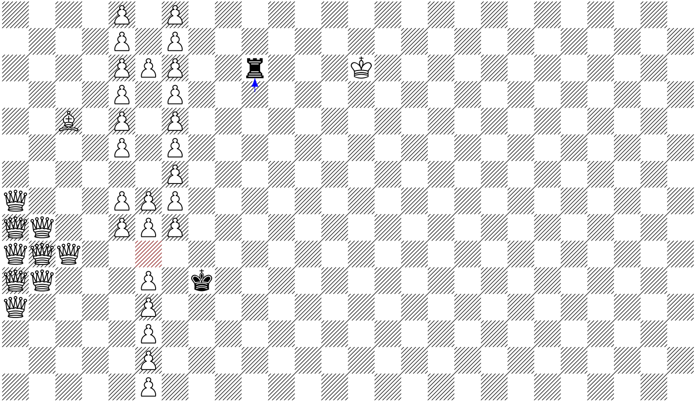
Releasing the Hordes, with value ω^2



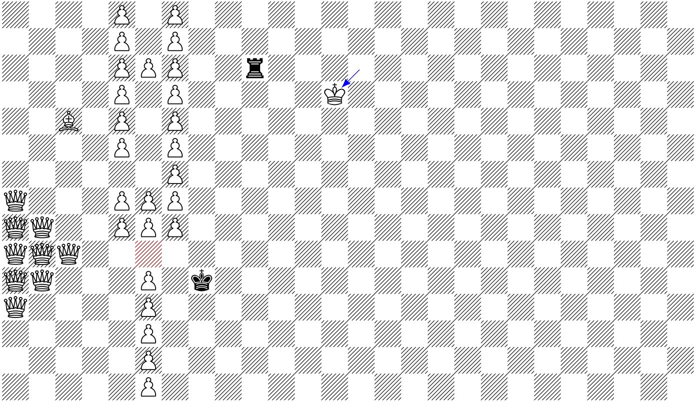
Releasing the Hordes, with value ω^2



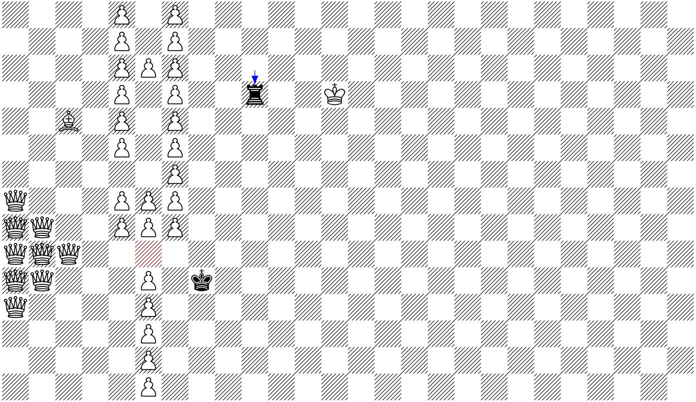
Releasing the Hordes, with value ω^2



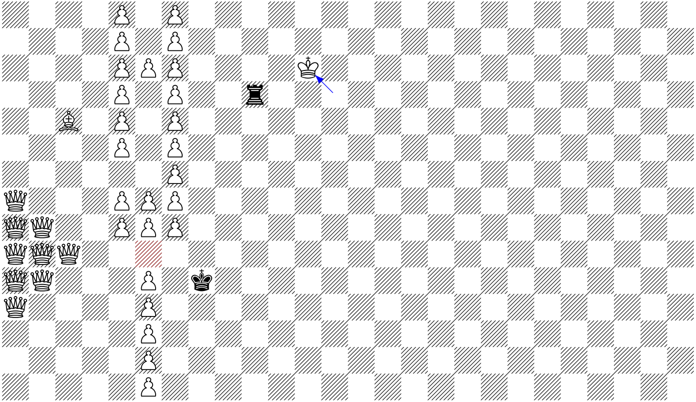
Releasing the Hordes, with value ω^2



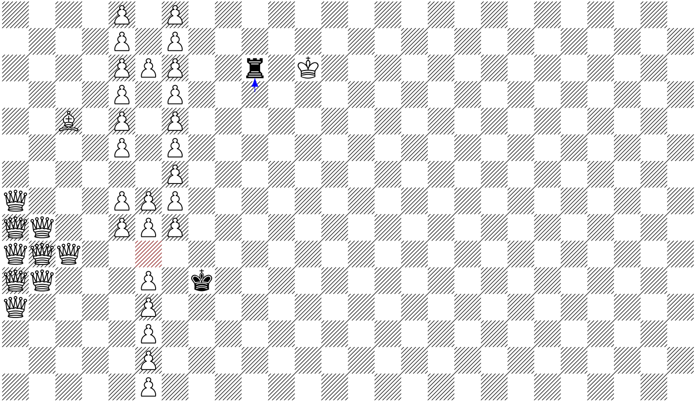
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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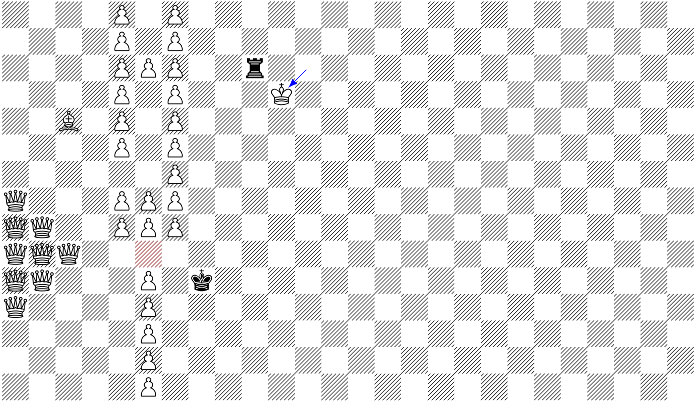

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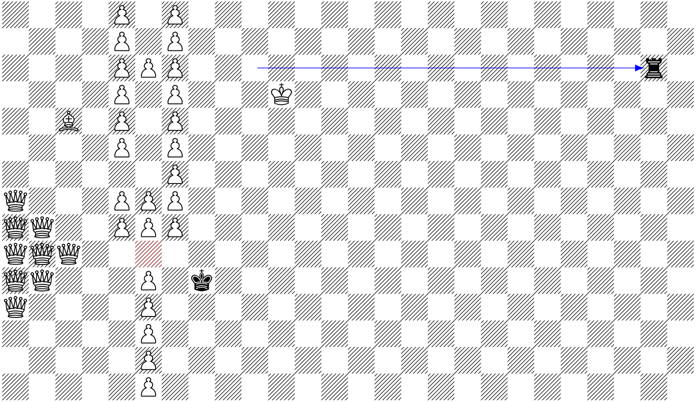

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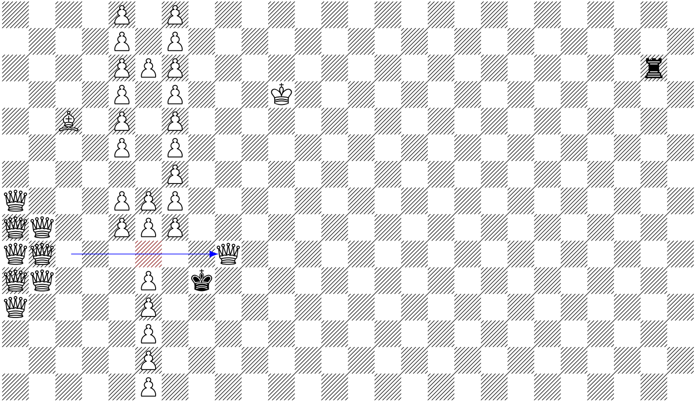
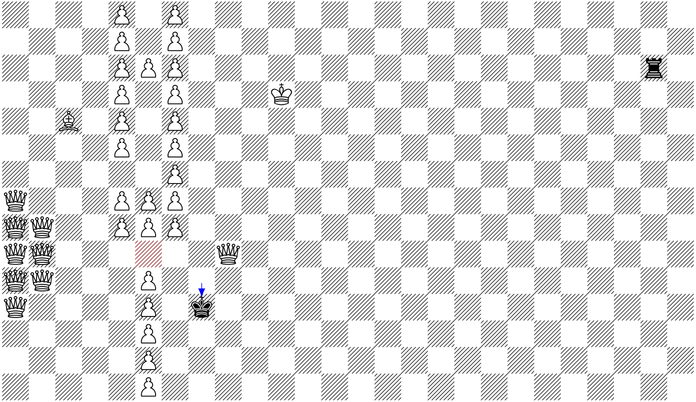


Queens enter the mating chamber.

Releasing the Hordes, with value ω^2



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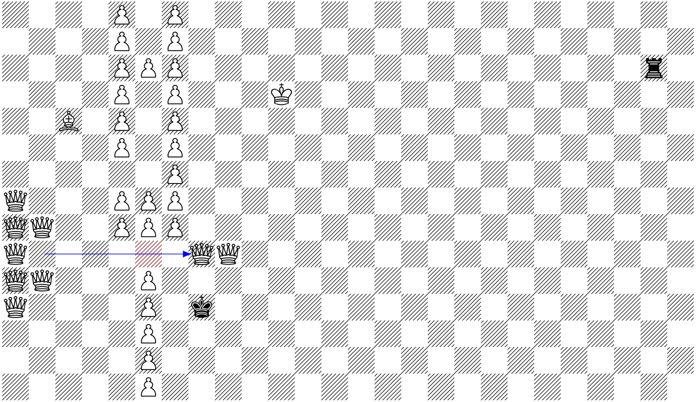
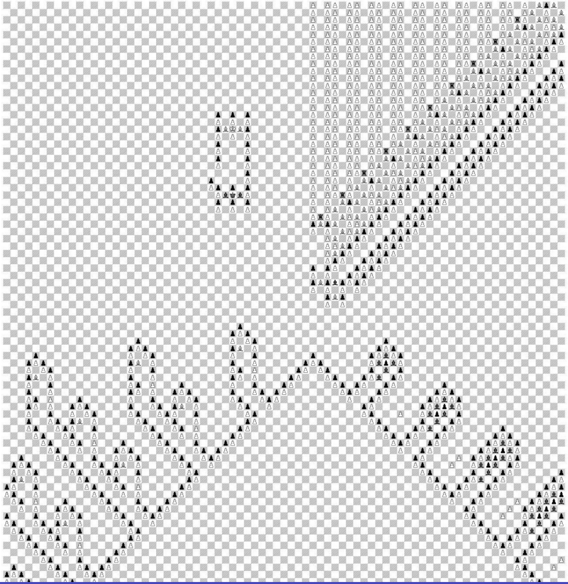
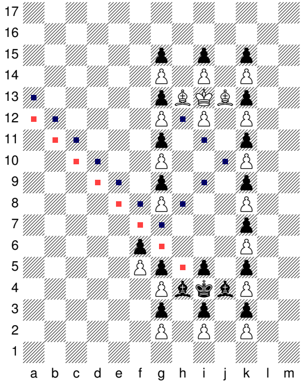


Checkmate

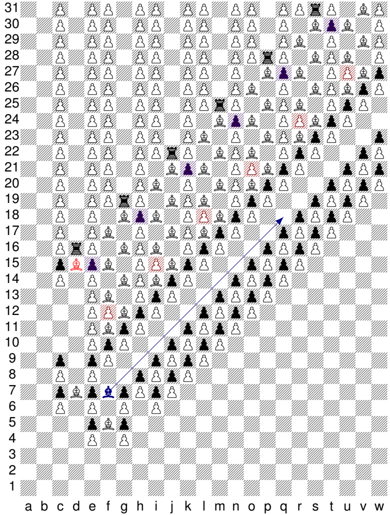
Previous state of the art: value ω^4



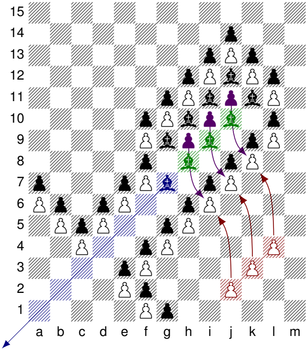
The throne room



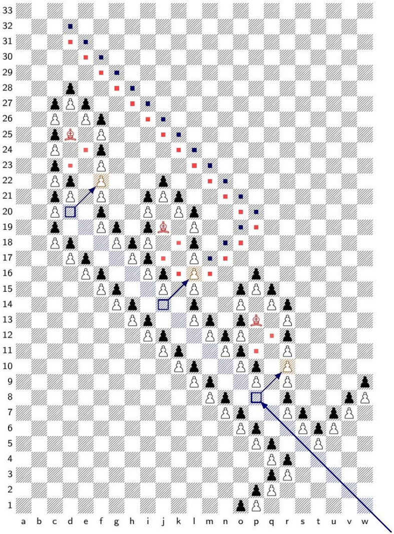
The rook towers



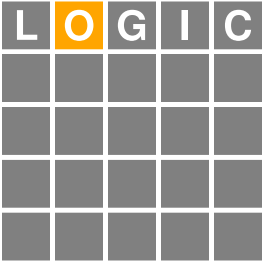
Bishop cannon



Bishop gateway terminal



Wordle



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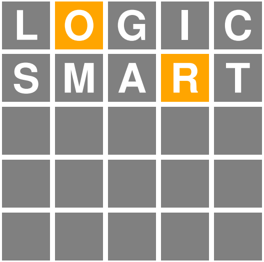

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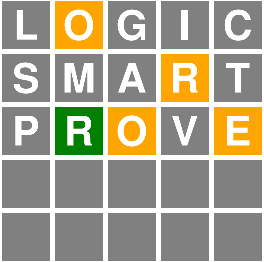

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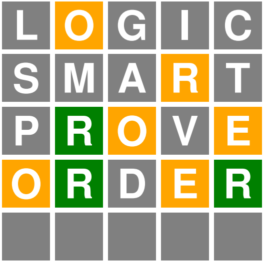

Wordle

Long Wordle

I should like to greatly enlarge the scale of the game.

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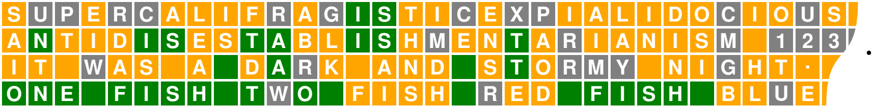

Consider a variation of Wordle using very long words or whole phrases, perhaps book-length passages—perhaps millions of letters!

Furthermore, let us consider any crazy fixed dictionary of allowed phrases, perhaps including many nonsense sequences of letters.

Nerdle

Consider, for example, the *Nerdle* variation.

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“Words” are now valid equations expressed using mathematical symbols:

$$1234567890+-\times/=$$

7	7	7	7	7	7	7	7	7	7	7	=	7	7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8	+	1	=	9	9	9	9	9	9	9	9	9	9
3	3	3	3	3	3	3	-	2	2	2	2	2	2	2	=	1	1	1	1	1	1	1
2	*	2	*	2	*	2	*	2	*	2	*	2	*	2	*	2	/	5	1	2	=	1

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2	*	2	*	2	*	2	*	2	*	2	*	2	*	2	*	2	/	5	1	2	=	1
5	1	2	/	2	/	2	/	2	/	2	/	2	/	2	/	2	/	2	/	2	=	1

Let’s consider Nerdle with extremely long equations, each using perhaps millions of symbols.

The Nerdle theorem

Question

In Nerdle with equations of length one million, how long will it take to win?

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Proof.

Start with any valid equation as first guess.

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Since 15 symbols, this can go on at most 15 steps. □

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Theorem (Hamkins [Ham23b])

In infinite Wordle with words of size ω and countable alphabet, codebreaker can win at stage ω.

Some dictionaries take that long—no strategy to win by finite stages.

Absurdle variation

Absurdle: two-player Wordle

The Absurdist gives feedback, but can change mind about the codeword, provided it remains consistent with previous feedback.

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Absurdle = worst-case play for codebreaker in Wordle

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The Absurdist wins with full dictionary on ω^ω—he can survive until stage ω, with still a codeword consistent with all feedback.

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Absurdist imagines true codeword is highly generic, using every letter exactly once, but rarely in the places guessed by the codebreaker. Gives gradually more specific finite information as play proceeds. Can survive until stage ω, when the word becomes known. □

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Corollary

With the full dictionary on ω^ω, there is no codebreaker strategy to win Wordle in finitely many moves.

The Wordle number

Definition

The *Wordle* number $\mathbb{w}$ is the size of the smallest dictionary $D \subseteq \omega^\omega$ that is not finitely winnable.

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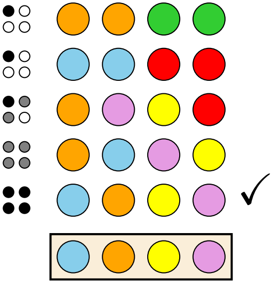
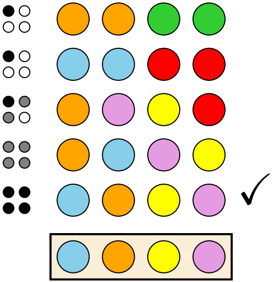
New result proved this morning (further salt)

$\mathbb{w} = \aleph_1$ always.

Mastermind

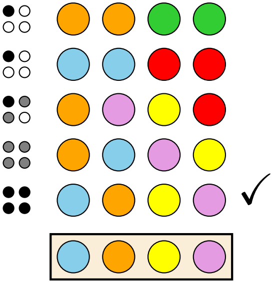


Mastermind



- A more subtle and difficult game

Mastermind



- A more subtle and difficult game
- Hidden codeword, codebreaker makes guesswords
- Mastermind feedback $(\kappa, \rho, \varepsilon)$

$\kappa =$ correctness count

$\rho =$ rearrangement count

$\varepsilon =$ incorrectness count

Infinite Mastermind

Theorem (Hamkins [Ham23b])

In infinite Mastermind with ω colors, duplication allowed, codebreaker can win at stage ω.

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In infinite Mastermind with ω colors, duplication allowed, codebreaker can win at stage ω.

Codebreaker poses the constant and nearly constant guesswords. From the response, one can deduce the true codeword.

Can't win by a countable ordinal stage

Theorem

In no-duplication infinite Mastermind, for no countable ordinal γ does codebreaker have a strategy to win by stage γ.

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For any countable γ, there is a generic codeword fulfilling all these promises. $\square$

In particular, there is no countable winning set of guesswords.

Mastermind number

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The *mastermind number* $\mathfrak{m}$ is the size of the smallest winning set for no-duplication infinite Mastermind using words of length ω over a countably infinite color set.

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Variant numbers for the game variants: $\mathbb{m}^* \quad \widehat{\mathbb{m}} \quad \mathbb{m}_{=,\neq} \quad \cdots$

Mastermind number is independent of ZFC

Theorem (Hamkins [Ham23b])

The mastermind number $\mathfrak{m}$ is an uncountable cardinal between $\aleph_1$ and the continuum $2^{\aleph_0}$.

$$\aleph_1 \quad \leq \quad \mathfrak{m} \quad \leq \quad 2^{\aleph_0}$$

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One can begin to see a connection with the covering number, by observing that having at least *n*-points of agreement with a given real is an open dense property. Being eventually different from a given sequence is meager.

Mastermind numbers

$$
\begin{array}{c}
\mathfrak{m}^* \\
\mathfrak{m}^{*,n} \\
\aleph_0
\end{array}
<
\begin{array}{c}
\widehat{\mathfrak{m}}^* \\
\widehat{\mathfrak{m}}^{*,n} \\
\omega+1
\end{array}
<
\begin{array}{c}
\mathfrak{w} \\
\aleph_1
\end{array}
\leq
\begin{array}{cc}
\mathfrak{m} & \widehat{\mathfrak{m}} \\
\mathfrak{m}_{=,\neq} & \widehat{\mathfrak{m}}_{=,\neq} \\
\mathfrak{m}_{=} & \widehat{\mathfrak{m}}_{=} \\
\mathfrak{m}^*_{=,\neq} & \widehat{\mathfrak{m}}^*_{=,\neq} \\
\mathfrak{m}^*_{=} & \widehat{\mathfrak{m}}^*_{=} \\
\multicolumn{2}{c}{\mathfrak{d}(\neq^*)} \\
\multicolumn{2}{c}{\mathfrak{d}(\neq^*, \overset{\omega}{\hookrightarrow}\omega)} \\
\multicolumn{2}{c}{\text{cov}(\mathcal{M})} \\
\multicolumn{2}{c}{\mathfrak{m}(\mathbb{C})}
\end{array}
\leq
\begin{array}{cc}
\mathfrak{m}_{\neq} & \widehat{\mathfrak{m}}_{\neq} \\
\mathfrak{m}^*_{\neq} & \widehat{\mathfrak{m}}^*_{\neq} \\
\mathfrak{m}^{*,n}_{\neq} & \widehat{\mathfrak{m}}^{*,n}_{\neq} \\
\mathfrak{m}^{*,n}_{=,\neq} & \widehat{\mathfrak{m}}^{*,n}_{=,\neq} \\
\mathfrak{m}^{*,n}_{=} & \widehat{\mathfrak{m}}^{*,n}_{=} \\
\multicolumn{2}{c}{\mathfrak{d}(\neq^*, {}^{\omega}n)} \\
\multicolumn{2}{c}{2^{\aleph_0}} \\
\multicolumn{2}{c}{\mathfrak{c}}
\end{array}
$$

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
University of Notre Dame

Credits

- Recursive chess image due to Django Pinter (Oxford)
- Chocolatier game analysis in another form essentially in [CL90].
- Infinite Hex and infinite Draughts were studied by Davide Leonessi in his Oxford MSc 2021 dissertation [Leo21]
- Some material adapted from [Ham20; Ham23a]

References I

[BHS12] Dan Brumleve, Joel David Hamkins, and Philipp Schlicht. "The Mate-in-*n* Problem of Infinite Chess Is Decidable". In: *How the World Computes*. Ed. by S. Barry Cooper, Anuj Dawar, and Benedikt Löwe. Vol. 7318. Lecture Notes in Computer Science. Springer, 2012, pp. 78–88. ISBN: 978-3-642-30869-7. DOI: 10.1007/978-3-642-30870-3_9. arXiv:1201.5597[math.LO]. http://wp.me/p5M0LV-f8.

[CL90] K Ciesielski and R. Laver. "A game of D. Gale in which one of the players has limited memory". *Period Math Hung* 21 (1990), pp. 153–158. DOI: 10.1007/BF01946852.

[EH14] C. D. A. Evans and Joel David Hamkins. "Transfinite game values in infinite chess". *Integers* 14 (2014), Paper No. G2, 36. ISSN: 1553-1732. arXiv:1302.4377[math.LO]. http://jdh.hamkins.org/game-values-in-infinite-chess.

References II

[EHP17] C. D. A. Evans, Joel David Hamkins, and Norman Lewis Perlmutter. "A position in infinite chess with game value ω^4". *Integers* 17 (2017), Paper No. G4, 22. arXiv:1510.08155[math.LO]. http://wp.me/p5M0LV-1c5.

[GS53] D. Gale and F. M. Stewart. "Infinite games with perfect information". *Ann. Math. Studies* 28 (1953), pp. 245–266.

[Ham20] Joel David Hamkins. *Proof and the Art of Mathematics*. MIT Press, 2020. ISBN: 978-0-262-53979-1. https://mitpress.mit.edu/books/proof-and-art-mathematics.

[Ham23a] Joel David Hamkins. *A Panorama of Logic*. book manuscript, 425 pages, in preparation, currently being serialized on https://www.infinitelymore.xyz/s/panorama-of-logic. 2023.

[Ham23b] Joel David Hamkins. "Infinite Wordle and the mastermind numbers". *Mathematics Logic Quarterly* (2023). DOI: 10.1002/malq.202200049. arXiv:2203.06804[math.LO]. http://jdh.hamkins.org/infinite-wordle-mastermind.

References III

[HL22] Joel David Hamkins and Davide Leonessi. “Transfinite game values in infinite draughts”. *Integers* 22 (2022). Paper G5, http://math.colgate.edu/~integers/wg5/wg5.pdf. arXiv:2111.02053[math.LO]. http://jdh.hamkins.org/transfinite-game-values-in-infinite-draughts.

[HL23] Joel David Hamkins and Davide Leonessi. “Infinite Hex is a draw”. *to appear in Integers* (2023). arXiv:2201.06475[math.LO]. http://jdh.hamkins.org/infinite-hex-is-a-draw.

[Lar10] Paul B. Larson. “Zermelo 1913”. In: *Ernst Zermelo - Collected Works/Gesammelte Werke: Volume I - Set Theory, Miscellanea / Band I - Mengenlehre, Varia*. Ed. by Heinz-Dieter Ebbinghaus, Craig G. Fraser, and Akihiro Kanamori. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 260–273. DOI: 10.1007/978-3-540-79384-7_9.

References IV

[Leo21] Davide Leonessi. “Transfinite game values in infinite games”. (2021). MSc dissertation, Mathematics and Foundations of Computer Science, University of Oxford. arXiv:2111.01630[math.LO]. http://wp.me/pdrDew-N.