Introduction to modal model theory

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This talk mainly includes joint work with:

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Paper is now finally appeared [HW24], arxiv preprint in 2020, blog post [Ham19].

See also prior work of Saveliev and Shapirovsky [SS16; SS18; SS20], on which this works overlaps in several matters independently.

Other prior/related work in [Ham03] [HW17] [HL08] [HL13] [HLL15] [HL22] [Ham18] [HW21] [BBL23]

Introducing modal model theory

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Define the modalities:

1. M thinks φ is *possible*, written $M \models \Diamond\varphi$, if there is an extension $M \sqsubseteq N$ with $N \models \varphi$.
2. M thinks φ is *necessary*, written $M \models \Box\varphi$, if every extension $M \sqsubseteq N$ has $N \models \varphi$.

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- All graphs
- All groups
- All fields
- Models of PA.
- Models of set theory.

Illustrating the modal vocabulary

Every graph thinks “possibly the diameter is 2.”

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Every graph thinks “possibly the diameter is 2.”

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Every field thinks possibly every element has a square root, but this is necessarily not necessary.

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$\Diamond\mathcal{L}$ assertions are substitution instances of $\mathcal{P}$ assertions $\varphi(p_0, \ldots, p_n)$ by $\mathcal{L}$ sentences:

$$\varphi(\psi_0, \ldots, \psi_n).$$

Remarkable expressive power of modal graph theory

The language of modal graph theory has a remarkable expressive power.

Let us illustrate this in several instances.

Theorem

2-colorability is expressible in modal graph theory.

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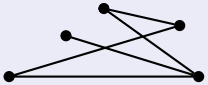
Proof.

G is 2-colorable $\iff$ possibly, there are adjacent nodes r and b, such that every node is adjacent to exactly one of them and adjacent nodes are connected to them oppositely.



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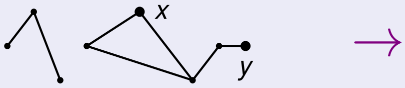

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$$\Box\forall c[(c \sim x \wedge \forall u, v\ (c \sim u \wedge u \sim v \wedge v \neq c \to c \sim v)) \to c \sim y].$$

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Finiteness is expressible in modal graph theory.

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G is finite $\iff$ possibly, there is n, whose neighbor graph is connected and all degree 2 except two vertices of degree 1, and all other nodes are adjacent to distinct neighbors of n.

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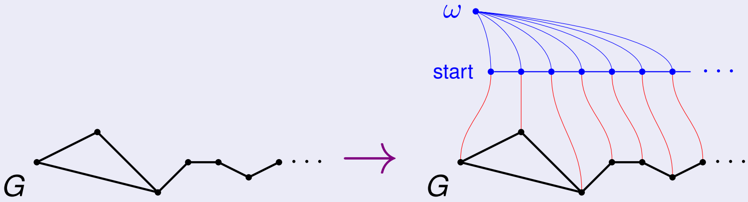

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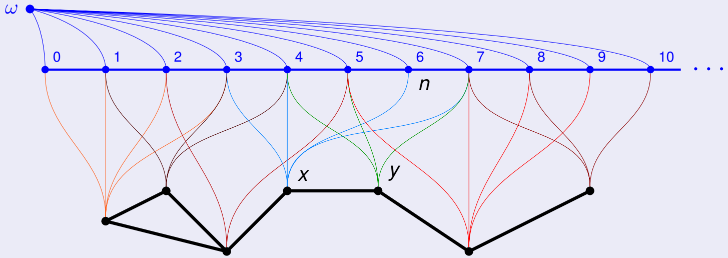

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It turns out that a large fragment of set-theoretic truth is interpretable in modal graph theory.

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We consider the class of all groups under the group extension relation.

See also [BBL23].

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But it is not expressible in the first-order language of group theory—take ultrapower of $\mathbb{Z}$.

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Being torsion is expressible in modal group theory

Torsion means every element has finite order.

Not expressible in language of first-order group theory.

Modal model theory

Let us now begin to develop some of the elementary modal model theory.

We focus on the case of $\mathrm{Mod}(T)$ for a fixed first-order theory T.

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Sam Adam-Day proved the two potentialist systems bisimilar.

$\mathcal{L}$ theory determines $\Diamond\mathcal{L}$ theory

Key Lemma

In $\mathrm{Mod}(T)$ for any first order theory

$$M \prec_{\mathcal{L}} N \qquad \text{if and only if} \qquad M \prec_{\Diamond\mathcal{L}} N.$$

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Lemma

$M \equiv_{\mathcal{L}} N$ if and only if $M \equiv_{\Diamond\mathcal{L}} N$.

(Appeared previously in work of Saveliev and Shapirovsky, [SS18, statement (5), p.17],[SS20, statement (5), p.1005])

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Open Question

Is every $\Diamond\mathcal{L}$ assertion equivalent to an assertion of $\mathcal{L}_{\omega_1,\omega}$?

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Modality elimination means that every modal assertion is equivalent to a modality-free assertion.

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For any first-order theory T, the following are equivalent:

1. *T admits modality trivialization over all assertions in $\mathcal{L}^{\Diamond}$.*
2. *T admits modality trivialization over all assertions in $\Diamond\mathcal{L}$.*
3. *T admits modality trivialization over all assertions in $\mathcal{L}$.*
4. *T is model complete.*

Modality trivialization

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Theory T is *model complete* if submodels $M \subseteq N$ are elementary $M \prec N$.

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$$\Diamond \exists x \forall y \, (x \sim y \leftrightarrow (@y \wedge @\forall z \, \neg y \sim z))$$

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Iterated semantics allow for a notion of relative actuality.

Modal graph theory with actuality

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Is actuality @ expressible in modal graph theory?

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- can interpret set-theoretic truth $\langle V, \in \rangle$.

Question

Is actuality @ expressible in modal graph theory?

The answer is no, confirming our conjecture.

We proved recently that @ sometimes allows you to define sets not definable without @.

Modal validities

A modal assertion $\varphi(p_1, \ldots, p_n)$ is *valid* at world M in potentialist system $\mathcal{W}$ for an allowed language if all substitution instances $\varphi(\psi_1, \ldots, \psi_n)$ arising for ψ_i in that language are true at M in $\mathcal{W}$.

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This is often sensitive to the allowed language of substitution instances, or whether parameters are allowed.

Easy lower bounds

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5. If $\mathcal{W}$ is linearly pre-ordered, then $\mathrm{S4.3}$ is valid with parameters.

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2. If buttons+switches, then validities contained in $\mathrm{S4.2}$.
3. If long ratchets+switches, then validities contained in $\mathrm{S4.3}$.
4. If railyards, then validities are exactly $\mathrm{S4}$.

Validating $\mathrm{S5}$

Theorem

1. *Every model in* $\mathrm{Mod}(T)$ *can be extended to one in which* $\mathrm{S5}$ *is valid for* $\mathcal{L}^{\Diamond}$ *sentences.*
2. *If* T *is* $\forall\exists$ *axiomatizable, then every model can be extended to one validating* $\mathrm{S5}$ *for* $\mathcal{L}^{\Diamond}$ *assertions with parameters.*

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Chains of models argument.

Theorem

A countable graph G validates S5 for $\mathcal{L}$ with parameters

$$\Diamond\Box\varphi(\bar{a}) \to \varphi(\bar{a})$$

Theorem

A countable graph G validates $\mathrm{S5}$ for $\mathcal{L}$ with parameters

$$\Diamond\Box\varphi(\bar{a}) \to \varphi(\bar{a})$$

if and only if G is the countable random graph.

Theorem

A countable graph G validates S5 *for* $\mathcal{L}$ *with parameters*

$$\Diamond\Box\varphi(\bar{a}) \to \varphi(\bar{a})$$

if and only if G is the countable random graph.

Theorem

A graph G validates S5 *for* φ *in* $\Diamond\mathcal{L}$ *with parameters iff it satisfies the theory of the countable random graph.*

Theorem

A countable graph G validates $\mathrm{S5}$ *for* $\mathcal{L}$ *with parameters*

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if and only if G is the countable random graph.

Theorem

A graph G validates $\mathrm{S5}$ *for* φ *in* $\Diamond\mathcal{L}$ *with parameters iff it satisfies the theory of the countable random graph.*

Theorem

G validates $\mathrm{S5}$ *for sentences iff G is universal for finite graphs.*

Validities in graphs

Theorem

Every graph G validates (for $\mathcal{L}$ assertions with parameters) either exactly $\mathrm{S4.2}$ *or exactly* $\mathrm{S5}$*.*

Validities in graphs

Theorem

Every graph G validates (for $\mathcal{L}$ assertions with parameters) either exactly $\mathrm{S4.2}$ or exactly $\mathrm{S5}$.

If it has the finite pattern property, get $\mathrm{S5}$. If not, there are independent buttons and switches, so $\mathrm{S4.2}$.

General case $\mathrm{Mod}(T)$

Theorem

A model $M \models T$ validates S5 for $\mathcal{L}$ with parameters

$$\Diamond\Box\varphi(\bar{a}) \to \varphi(\bar{a})$$

General case $\mathrm{Mod}(T)$

Theorem

A model $M \models T$ validates $\mathrm{S5}$ for $\mathcal{L}$ with parameters

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if and only if M is existentially closed in $\mathrm{Mod}(T)$.

General case $\mathrm{Mod}(T)$

Theorem

A model $M \models T$ validates $\mathrm{S5}$ for $\mathcal{L}$ with parameters

$$\Diamond\Box\varphi(\bar{a}) \to \varphi(\bar{a})$$

if and only if M is existentially closed in $\mathrm{Mod}(T)$.

This result explains what was important about the countable random graph.

Universal S5 is impossible

Theorem

If every model in $\mathrm{Mod}(T)$ *validates* S5 *for* $\mathcal{L}$ *assertions with parameters, then* T *is model complete and consequently admits modality trivialization.*

Universal $\mathrm{S5}$ is impossible

Theorem

If every model in $\mathrm{Mod}(T)$ *validates* $\mathrm{S5}$ *for* $\mathcal{L}$ *assertions with parameters, then* T *is model complete and consequently admits modality trivialization.*

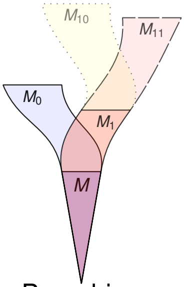
So $p \leftrightarrow \Diamond p$ also is valid, and this is not part of $\mathrm{S5}$.

Conclusion

The validities of $\mathrm{Mod}(T)$ cannot be *exactly* $\mathrm{S5}$.

Varieties of potentialism

The modal language enables us to express sweeping general principles describing the nature of our potentialist conception.



Linear inevitability S4.3

Directed convergence S4.2

Branching possibility S4

Thank you.

Article is now available:

[HW24] Joel David Hamkins and Wojciech Aleksander Wołoszyn, “Modal model theory,” Notre Dame Journal of Formal Logic, 65:1(2024). http://jdh.hamkins.org/modal-model-theory.

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