

Introduction to modal model theory

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame
VRF, Mathematical Institute, Oxford

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This talk includes joint work with:

Wojciech Aleksander Wołoszyn, Oxford University

Paper is available [HW24].

Introducing modal model theory

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A *potentialist system* is a class of models \mathcal{W} with an extension relation $M \sqsubseteq N$, refining the substructure relation.

Define the modalities:

- 1 M thinks φ is *possible*, written $M \models \Diamond \varphi$, if there is an extension $M \sqsubseteq N$ with $N \models \varphi$.
- 2 M thinks φ is *necessary*, written $M \models \Box \varphi$, if every extension $M \sqsubseteq N$ has $N \models \varphi$.

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- All fields
- Models of PA.
- Models of set theory.

Illustrating the modal vocabulary

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Every field thinks possibly every element has a square root, but this is necessarily not necessary.

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$\diamond \mathcal{L}$ assertions are substitution instances of \mathcal{P} assertions $\varphi(p_0, \dots, p_n)$ by \mathcal{L} sentences:

$$\varphi(\psi_0, \dots, \psi_n).$$

Remarkable expressive power of modal graph theory

The language of modal graph theory has a remarkable expressive power.

Let us illustrate this in several instances.

Theorem

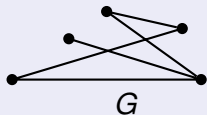
2-colorability is expressible in modal graph theory.

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Proof.

G is 2-colorable \iff possibly, there are adjacent nodes r and b , such that every node is adjacent to exactly one of them and adjacent nodes are connected to them oppositely.

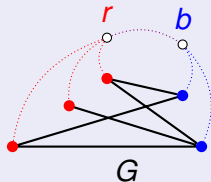
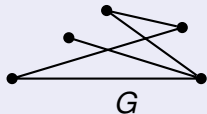


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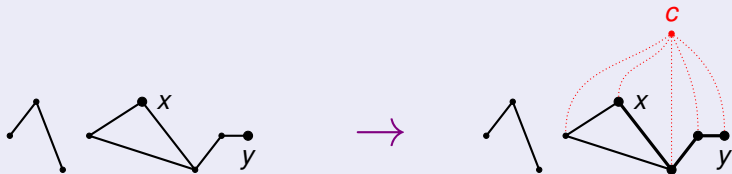


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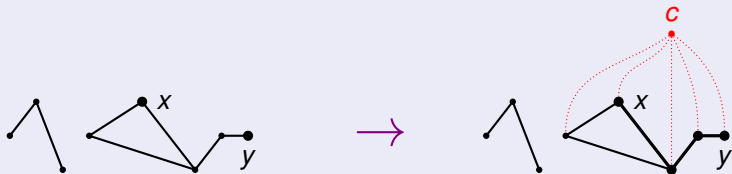


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$$\square \forall c [(c \sim x \wedge \forall u, v (c \sim u \wedge u \sim v \wedge v \neq c \rightarrow c \sim v)) \rightarrow c \sim y].$$

□

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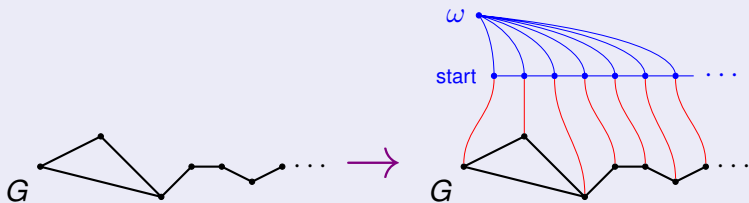
G is countable \iff possibly, there is ω , with neighbor graph connected and all of degree 2 except one node, and all other nodes adjacent to distinct neighbors of ω .

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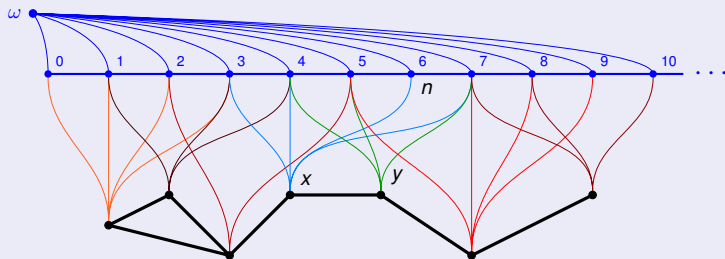
G has size at most continuum \iff if we can associate every node in the graph with a distinct subset of ω .

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It turns out that a large fragment of set-theoretic truth is interpretable in modal graph theory.

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We consider the class of all groups under the group extension relation.

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But it is not expressible in the first-order language of group theory—take ultrapower of \mathbb{Z} .

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Being torsion is expressible in modal group theory

Torsion means every element has finite order.

Not expressible in language of first-order group theory.

Modal model theory

Let us now begin to develop some of the elementary modal model theory.

We focus on the case of $\text{Mod}(T)$ for a fixed first-order theory T .

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Sam Adam-Day proved the two potentialist systems bisimilar.

\mathcal{L} theory determines $\diamond \mathcal{L}$ theory

Key Lemma

In $\text{Mod}(T)$ for any first order theory

$$M \prec_{\mathcal{L}} N \quad \text{if and only if} \quad M \prec_{\diamond \mathcal{L}} N.$$

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Open Question

Is every $\diamond \mathcal{L}$ assertion equivalent to an assertion of $\mathcal{L}_{\omega_1, \omega}$?

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Modality elimination means that every modal assertion is equivalent to a modality-free assertion.

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- 1** *T admits modality trivialization over all assertions in \mathcal{L}^\Diamond .*
- 2** *T admits modality trivialization over all assertions in $\Diamond\mathcal{L}$.*
- 3** *T admits modality trivialization over all assertions in \mathcal{L} .*
- 4** *T is model complete.*

Modality trivialization

Modality trivialization means $\Diamond\varphi$ is equivalent to φ .

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Theory T is *model complete* if submodels $M \subseteq N$ are elementary $M \prec N$.

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$$\diamond \exists x \forall y (x \sim y \leftrightarrow (@y \wedge @\forall z \neg y \sim z))$$

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Iterated semantics allow for a notion of relative actuality.

Modal graph theory with actuality

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Is actuality @ expressible in modal graph theory?

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- can interpret set-theoretic truth $\langle V, \in \rangle$.

Question

Is actuality @ expressible in modal graph theory?

The answer is no, confirming our conjecture.

We proved recently that @ sometimes allows you to define sets not definable without @.

Modal validities

A modal assertion $\varphi(p_1, \dots, p_n)$ is *valid* at world M in potentialist system \mathcal{W} for an allowed language if all substitution instances $\varphi(\psi_1, \dots, \psi_n)$ arising for ψ_i in that language are true at M in \mathcal{W} .

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This is often sensitive to the allowed language of substitution instances, or whether parameters are allowed.

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- 5 If \mathcal{W} is linearly pre-ordered, then S4.3 is valid with parameters.

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- 4 If railyards, then validities are exactly S4.

Validating S5

Theorem

- 1 *Every model in $\text{Mod}(T)$ can be extended to one in which S5 is valid for \mathcal{L}^\diamond sentences.*
- 2 *If T is $\forall\exists$ axiomatizable, then every model can be extended to one validating S5 for \mathcal{L}^\diamond assertions with parameters.*

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Chains of models argument.

Theorem

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Theorem

G validates S5 for sentences iff G is universal for finite graphs.

Validities in graphs

Theorem

Every graph G validates (for \mathcal{L} assertions with parameters) either exactly S4.2 or exactly S5.

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If it has the finite pattern property, get S5. If not, there are independent buttons and switches, so S4.2.

General case $\text{Mod}(T)$

Theorem

A model $M \models T$ validates S5 for \mathcal{L} with parameters

$$\diamond \Box \varphi(\bar{a}) \rightarrow \varphi(\bar{a})$$

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This result explains what was important about the countable random graph.

Universal S5 is impossible

Theorem

If every model in $\text{Mod}(T)$ validates S5 for \mathcal{L} assertions with parameters, then T is model complete and consequently admits modality trivialization.

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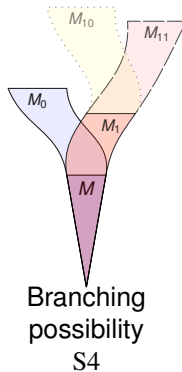
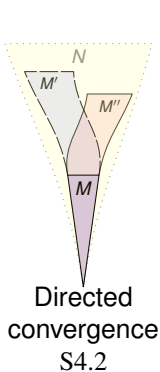
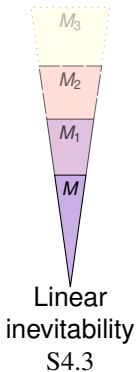
So $p \leftrightarrow \Diamond p$ also is valid, and this is not part of S5.

Conclusion

The validities of $\text{Mod}(T)$ cannot be *exactly* S5.

Varieties of potentialism

The modal language enables us to express sweeping general principles describing the nature of our potentialist conception.



Thank you.

Article is now available:

[HW24] Joel David Hamkins and Wojciech Aleksander Wołoszyn, “Modal model theory,” *Notre Dame Journal of Formal Logic*, 65:1(2024). <http://jdh.hamkins.org/modal-model-theory>.

Joel David Hamkins
Professor of Logic
Notre Dame

References

- [HW24] Joel David Hamkins and Wojciech Aleksander Wołoszyn. “Modal model theory”. *Notre Dame Journal of Formal Logic* 65.1 (2024), pp. 1–37. DOI: [10.1215/00294527-2024-0001](https://doi.org/10.1215/00294527-2024-0001). [arXiv:2009.09394](https://arxiv.org/abs/2009.09394) [math.LO].