Pluralism in the foundations of mathematics

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The theme

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We shall get into the debate on pluralism versus monism in the foundations of mathematics.

Set theory as a foundation of mathematics

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Set theory can interpret essentially arbitrary mathematical structure—diverse mathematical structures are faitfully interpreted in set theory.

In this way, set theory provides a generous arena, a single arena in which one can view all mathematics taking place.

The Set-Theoretical Universe

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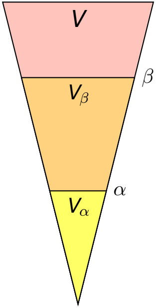

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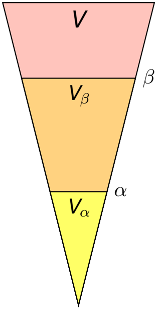

Starting from nothing, sets accumulate in a well-founded transfinite hierarchy, iterating the power set.

The cumulative universe seems to be the intended realm of set theory.

The Universe View

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On the universe view, questions in set theory have definite answers: axiom of choice, continuum hypothesis, existence of large cardinals. There will be facts of the matter.

Thus, the universe view is one of determinism for set-theoretic truth, and hence also determinism for mathematical truth.

Main challenge for the universe view

A difficulty for the Universe view. The central discovery in set theory over the past half-century has been the enormous range of set-theoretic possibility.

The most powerful set-theoretical tools are most naturally understood as methods of constructing alternative set-theoretical universes, universes that seem fundamentally set-theoretic.

forcing, ultrapowers, canonical inner models, etc.

Much of set-theory research has been about constructing as many different models of set theory as possible, exhibiting precise features or relations to other models.

An imaginary alternative history

Imagine that set theory had followed a different history:

- Imagine that as set theory developed, theorems were increasingly settled in the base theory.
- ...that the independence phenomenon was limited to paradoxical-seeming meta-logic statements.
- ...that the few true independence results occurring were settled by missing natural self-evident set principles.
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Such developments could have constituted evidence for the Universe view.

But the actual history is not like this.

Actual history: an abundance of universes

Over the past half-century, set theorists have discovered a vast diversity of models of set theory, a chaotic jumble of set-theoretic possibilities.

Whole parts of set theory exhaustively explore the combinations of statements realized in models of set theory, and study the methods supporting this exploration.

Set theorists build models to order.

Set-theoretic pluralism

A competing philosophical position accepts the alternative set concepts as fully real.

Set-theoretic pluralism. The philosophical position holding that there are many different legitimate concepts of set, each giving rise to a corresponding set-theoretic universe.

Also known as the *multiverse view*.

Set-theoretic pluralism can be realist

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There has been a subtle change over the years in what it means to describe oneself as a “platonist”. Platonism is and should be about the real existence of the objects, rather than about the uniqueness of the set-theoretic realm.

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- Forcing reveals the ubiquity of the independence phenomenon—almost every nontrivial question in set theory, it turns out, is independent of the relevant theory.
- Forcing extensions $V \subseteq V[G]$ seem totally acceptable set-theoretically.
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- Forcing extensions $V \subseteq V[G]$ seem totally acceptable set-theoretically.
- Forcing admits a richly mathematical metatheory.
- Forcing extensions arise exactly by using a multi-valued concept of set in the cumulative hierarchy $V^{\mathbb{B}}$.

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The “dream solution” to CH is now impossible.

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But this relies on monist conception of ordinals as “completed” totality.

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- L has same consistency assertions as V.
- L has transitive models of same large cardinal theories.
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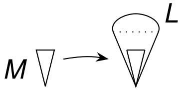

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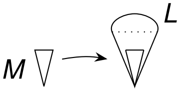

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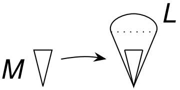

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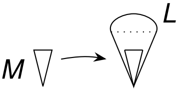
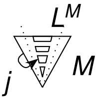


- (Hamkins,Williams) Can resurrect truths of any inner model.
- (HW [HW21]) Can make the extension pointwise definable.
- (Hamkins [Ham13]) Every countable M is isomorphic to a submodel of its own constructible universe $j : M \to L^M$.



$$x \in y \longleftrightarrow j(x) \in j(y)$$

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That is great, but this is a potentialist move, hence pluralist.

Opens the question: what does it mean to “adopt” a theory?

Enlarging the object-theory/metatheory distinction

Pluralism tends to enlarge the object-theory/meta-theory distinction.

There is not just one metatheory, but a vast hierarchy of metatheories.

Every sufficient theory can serve as a metatheory for the models and theories available to it, and serves as an object theory for the theories above it.

In particular, under pluralism, the metatheory also is pluralist.

What is a philosophy for?

There is essentially nothing mathematical at stake in the dispute between set-theoretic monism and pluralism.

Rather, the debate is about: where should set theory go?

The two perspectives seem to suggest certain avenues of investigation as interesting or fruitful.

From this point of view, we might judge the two positions by the mathematics they have inspired.

The modal logic of forcing

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- View models of set theory as possible worlds.
- A model M can access its forcing extensions $M[G]$.
- Gives rise to the forcing modalities $\Diamond, \Box$
- What are the validities?
- A rich literature of work. [HL08; HL13; HLL15]

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- Forcing potentialism
- Γ-forcing potentialism, for various Γ
- Countable transitive model potentialism
- Top-extensional potentialism
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We've worked out the modal commitments for each form of potentialism. [HL22; HW17; HW21]

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New work: modal model theory, category-theoretic potentialism, graphs, groups, orders, etc.

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- (FHR) The Mantle is the intersection of all grounds.
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Fascinating development (Usuba): if there is an extendible cardinal, then there is a bedrock.

As a result, the large-cardinal set theorists are making use of the geology theory—the results connect pluralism with monism.

A thought experiment about CH

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I shall identify contingency in our judgments on whether a principle is fundamental.

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Incipient forms of the transfer principle and saturation.

- “The two number realms fulfill the same basic truths.”
- “Every imaginable gap in $\mathbb{R}$ is filled by infinitesimals.”

Such ideas could lead to a robust pre-rigorous infinitesimal theory of calculus. Not much different from our reality.

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- ZFC+CH proves there is a unique saturated field extension of size continuum with the transfer principle. A back-and-forth argument generalizing Cantor's account of $\mathbb{Q}$.
- We know that this is not possible in ZFC without CH. "The hyperreals" is not meaningful in ZFC, a key explanation, in my view, of hesitancy about NSA.

But hyperreal field $\mathbb{R}^*$ admits a categorical account in ZFC+CH.

How CH gets on the axiom list

The thought experiment, at bottom, is that the hyperreals $\mathbb{R}^*$ are long a core mathematical idea, pre-rigorous at first, but then with a rigorous categorical account.

To provide the categorical account, the Zermelo-like figure provides a list of axioms, something like ZFC+CH.

We know that this is possible and that CH cannot be omitted.

So this is how CH gets onto the list of fundamental axioms—these would be the axioms that make sense of calculus, by providing the foundations of $\mathbb{R}$ and $\mathbb{R}^*$.

The later discovery by forcing that CH is required would only strengthen the judgment.

Justifying CH

Extrinsic justification of CH

CH would be seen as a fundamental and necessary principle for mathematics, making sense of core mathematical ideas in calculus regarding the coherence of $\mathbb{R}^*$.

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Possible intrinsic justification of CH

After extrinsic justification is successful, I imagine a possible intrinsic justification finding appeal. Namely, CH asserts that the two methods of achieving uncountability agree. A unifying, explanatory principle of the uncountable.

Further support and effects

In the thought-experiment world, the later discovery by Gödel that CH holds in L would greatly support the attitude, and could even lend to extrinsic support for $V = L$.

The GCH could find extrinsic support by giving rise to categorical accounts of higher-cardinality analogues of the hyperreals.

The discovery via forcing that without CH there can be multiple non-isomorphic $\mathbb{R}^*$ would be seen as chaotic/bizarre, much like current attitudes about the weird $\neg\mathrm{AC}$ models, with amorphous sets, non-unique $\bar{\mathbb{Q}}$, partitions of $\mathbb{R}$ with more than continuum many classes, etc.

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Thus, there is contingency in our attitudes toward CH.

Geometry

For two thousand years, geometry studied concepts—points, lines, planes—with a seemingly clear, absolute meaning.

The singularity of those fundamental concepts, however, shattered via non-Euclidean geometry into distinct geometrical concepts, realized in distinct geometrical universes.

Thus, we have geometric pluralism.

Geometric pluralism

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- Geometry shows pluralism need not reduce to formalism.
- Geometry shows that pluralism is not a form of skepticism.
- Geometry shows that pluralism is not disqualifying for a foundational theory.

Helpful to consider the analogy with geometry in face of objections to set-theoretic pluralism.

Arithmetic pluralism

Let me turn now to discuss a case of pluralism that many mathematicians find a bit incredible.

Namely, the case of pluralism in arithmetic foundations.

Arithmetic definiteness

To be sure, many mathematicians take there to be facts of the matter about arithmetic assertions.

The question whether there are infinitely many prime pairs, for example, is seen as having a definite answer, albeit one not yet known.

Similarly, the busy-beaver function BB(n) is thought to have definite "true" values, even when these are independent of strong theories ZFC $+ \mathrm{LC}$. And for the Riemann hypothesis, Goldbach's conjecture, and indeed any arithmetic assertion, there is a fact of the matter about whether it is true or false.

This view is also known as arithmetic monism, opposed to arithmetic pluralism.

Two kinds of arithmetic definiteness

Semantic definiteness

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Perhaps semantic definiteness for arithmetic is grounded in a more fundamental ontological definiteness:

Ontological definiteness of $\mathbb{N}$

There is a definite singular structure of arithmetic, with numbers

$$0 \quad 1 \quad 2 \quad 3 \quad \text{and so on} \ldots$$

and operations of successor, addition, multiplication.

The definiteness of the arithmetic structure would explain the definiteness of the arithmetic theory.

What are the arguments for arithmetic definiteness?

Why are mathematicians so confident they have a robust singular conception of the natural numbers?

Do we have a definite singular conception of what it means to be finite?

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C'mon, it is obvious

Many mathematicians simply find it intuitively obvious that there is a definite structure of the natural numbers.

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But of course, “and so on” cannot carry the weight of philosophical argument. One must say more.

What are the actual arguments for the definiteness of our concept of the finite?

Multiple definitions of the finite

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- Dedekind finite
- Numerically finite, via Dedekind categoricity
- Frege's definition via the anscestral
- Tarski's definitions
- Stäckel finite

The categoricity argument

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Dedekind proved definitions by recursion are legitimate, and in this way we get the rest of the familiar arithmetic structure.

Categoricity: any two models $\langle\mathbb{N}, 0, S\rangle$ and $\langle\bar{\mathbb{N}}, \bar{0}, \bar{S}\rangle$ are isomorphic. Define the isomorphism by recursion.

Seems to provide reason for confidence that there is a unique natural number structure.

Categoricity

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Can also be viewed as taking place in second-order logic.

Does this settle the issue? Does it provide reason to think we have a definite singular conception of the finite?

Not really. If we doubt the definiteness of our concept of the finite, then surely we doubt just as much the definiteness of higher realms of set theory or second-order logic.

Seems circular to establish definiteness of the finite, presuming definiteness in the comparatively murky realm of arbitrary sets.

ZFC arithmetic

We already know that different models of ZFC can have different non-isomorphic arithmetic structures.

After all, by Gödel's theorem the arithmetic theorems of ZFC must be incomplete.

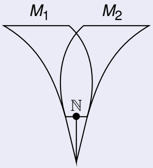
So the various "standard" models of arithmetic arising as $\mathbb{N}^M$ in a model of set theory M can realize different theories.

They can disagree on specific values of BB(n), on consistency assertions, on whether a given polynomial equation has integer solutions.

Definiteness of objects $\not\Rightarrow$ Definiteness of truth

Hamkins & Yang [HY14]

There are models of set theory that agree on the structure of the natural numbers, but disagree on arithmetic truth.



$M_1, M_2 \models \mathrm{ZFC}$

$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$

M_1 believes $\mathbb{N} \models \sigma$

M_2 believes $\mathbb{N} \models \neg\sigma$

Similar phenomenon with $\mathbb{R}$ and projective truth, H_{ω_2} and so on.

Conclusion: definiteness of truth does not flow from definiteness of structure. It is a strictly additional metatheoretic commitment.

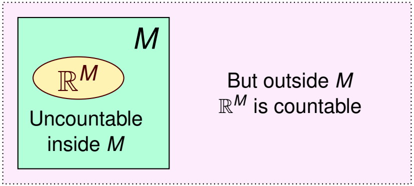
Disagreement on the concept of finite

We already know how it is possible for different metatheoretic contexts to have different concepts of the finite.

Let us explore how that can happen.

The Skolem paradox

Skolem observed that a model of set theory M can be wrong in its judgment that a set is uncountable.

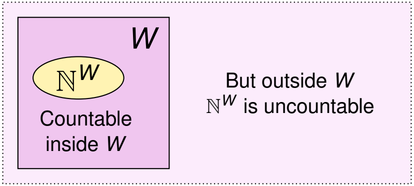


If M is any countable model of ZFC, then the set of real numbers $\mathbb{R}^M$ is uncountable from M's perspective, but countable outside M.

Judgments that a set is countable or uncountable can thus depend on the set-theoretic background in which they are made.

Reverse Skolem paradox

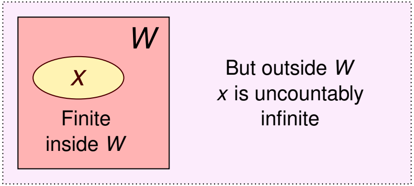
We can also arrange the reverse situation, where a set is countable inside a model W, but uncountable outside.



For example, take the theory ZFC with uncountably many constants n_α, asserting they are pairwise distinct natural numbers. By compactness, there is a model.

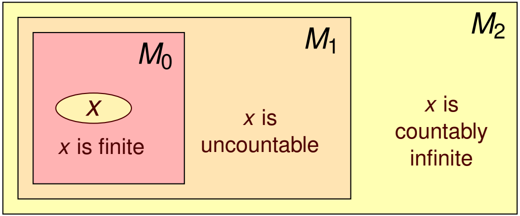
Extreme reverse Skolem paradox

By pushing this harder, we can have all $n_\alpha < x$ for some natural number x in W.



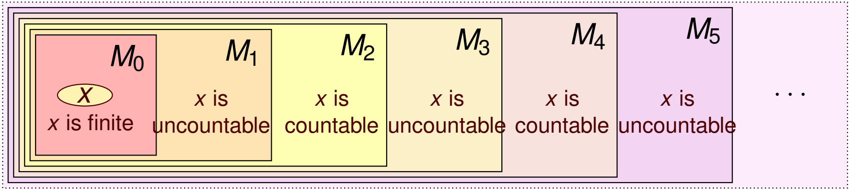
Iterated nonabsoluteness

There are models $M_0 \subseteq M_1 \subseteq M_2$ of ZFC viewing a set x as finite, then uncountable, and finally countably infinite:



Indefinitely iterated indefiniteness

Indeed, we can extend the pattern of indefiniteness indefinitely.



Finiteness is downwards absolute from any metatheoretic context to the models available there.

A rich hierarchy of arithmetic contexts

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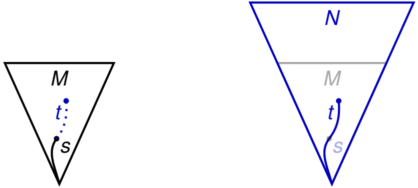
End-extensional arithmetic potentialism [Ham18].



- Universal algorithm
- Woodin's argument for determinism and free will
- Sheds light on ultrafinitism

A curious case of interpretation

The theory $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ is often regarded as foundationally crippled, perhaps irredeemably flawed.

A curious case of interpretation

The theory $\mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA})$ is often regarded as foundationally crippled, perhaps irredeemably flawed.

Yet I recently observed:

Theorem

There is a sentences σ such that $\mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA}) + \sigma$ is equiconsistent with ZFC*, and similarly with any extension of* ZFC*.*

Thus, $\mathrm{PA} + \neg\,\mathrm{Con}(\mathrm{PA})$ can still serve as a foundational theory.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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