The math tea argument—must there be numbers we cannot describe or define?

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The math tea argument

Heard at a good math tea anywhere:

“There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions.”

Does this argument withstand scrutiny?
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“I can describe any number. Let me show you: you tell me a number, and I’ll tell you a description of it.”

–Horatio, age 8
Definability

An object $r$ is *definable* in a structure $\mathcal{M}$ if it is the unique object in that structure satisfying some assertion.

$$\mathcal{M} \models \varphi[x] \iff x = r.$$
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\mathcal{M} \models \varphi[x] \iff x = r.
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A definable object has a property in a structure that only it has.
| Definability and the Math Tea argument, Pavia 2022 | Joel David Hamkins |

### Definability

#### Real continuum $\langle \mathbb{R}, < \rangle$

No point is definable, since any two real numbers are automorphic by translation.
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The number 0 is definable, since it is the only additive idempotent

$$z = 0 \iff \langle \mathbb{Z}, + \rangle \models z + z = z.$$
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No other elements are definable, because negation $x \mapsto -x$ is an automorphism.
Pointwise definability

Ring of integers $\langle \mathbb{Z}, +, \cdot \rangle$

The number 1 is the unique multiplicative identity.
**Pointwise definability**

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Every integer is definable in this structure.
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We can then define 2 as \( 1 + 1 \) and \(-2\) as the additive inverse, and so on.

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Thus, \( \langle \mathbb{Z}, +, \cdot \rangle \) is pointwise definable: every individual is definable.
Ordered real field $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$

Note that the order $<$ is definable from algebraic structure

$$x < y \iff \exists a \neq 0 \quad x + a^2 = y.$$
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- Every algebraic number is definable.
Ordered real field

But only algebraic numbers are definable in \( \langle \mathbb{R}, +, \cdot, 0, 1, < \rangle \).

**Theorem (Tarski)**

*In the ordered real field* \( \langle \mathbb{R}, +, \cdot, 0, 1 \rangle \), *every formula* \( \varphi(x) \) *is equivalent to a quantifier-free formula.*

One begins to see this by recalling

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\exists x \ ax^2 + bx + c = 0 \iff b^2 - 4ac \geq 0.
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**Corollary**

*The field of real algebraic numbers* \( \mathbb{A} \) *is an elementary substructure of* \( \langle \mathbb{R}, +, \cdot, 0, 1, < \rangle \).*
Corollary

*There is a computable procedure to decide truth of any statement in* $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$.

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*Cartesian geometry (in any finite dimension) is decidable.*

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The algebraic numbers are a real-closed field. So the elimination result works in the algebraic numbers. So $\mathbb{A}$ is an elementary substructure. So every definable element is algebraic.
Leibnizian models

A model $M$ is Leibnizian if any two distinct points have different properties: if $a \neq b$, then there is some formula $\varphi$ such that

$$M \models \varphi(a) \land \neg \varphi(b).$$
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**Question**

Are these notions the same?
Leibnizian vs. pointwise definable models

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This is a successful instance of the Math Tea argument.
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Note that \( \langle \mathbb{Z}, <, A \rangle \) is rigid, even though it has no definable elements.
More structure

As we add structure, we can define more real numbers.
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Can now define $\pi$. 
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In particular, every computable real number and much more is definable.
Computable numbers

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Problem: given programs for $a$, $b$, we cannot compute digits of $a + b$.

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\begin{align*}
a &= 0.343434343434 \ldots \\
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Better idea: use rational approximations to $z$, to within a specified accuracy. Very robust conception.
More structure, more context $\rightarrow$ more definability

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Consider the real $0.110101110 \ldots$, where $n$ bit is 1, if the generalized continuum hypothesis holds at $\aleph_n$, otherwise 0.
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In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.
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- or to define objects in \( \langle V_\alpha, \in \rangle \) when \( \alpha \) is not itself definable. (This amounts to using \( \alpha \) as a parameter.)

We are thereby pushed:

- to allow only countable languages, and
- to consider only structures that are themselves definable with respect to the set-theoretic background \( \langle V, \in \rangle \).
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Does it withstand scrutiny?

Well, it’s complicated.
In a fixed structure

In a fixed set-sized structure \( \mathcal{M} \) in a countable language and with all the real numbers in it, the math tea argument is fine: there are only countably many definitions, but uncountably many reals.

We simply associate each definable object \( r \) with a formula \( \psi_r \) that defines it. With access to such a definability map

\[
\psi_r \mapsto r,
\]

we may diagonalize against it to produce a real that is not definable.
Meta-mathematical obstacle

When defining reals $r$ over the full set-theoretic universe $\langle V, \in \rangle$, however, a subtle meta-mathematical obstacle arises:

The property of being definable in $\langle V, \in \rangle$ is not first-order expressible in set theory.
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The key subtlety is that if we lack the association of definition with object defined, we cannot undertake the diagonalization to produce the non-definable real.
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By Tarski’s theorem on the nondefinability of truth, this is impossible.
Metamathematical issues for the Math Tea argument

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We might expect that in any model of ZFC, there must be real numbers that are not definable in that model.

But that isn’t true.
Pointwise definable models of set theory

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**Theorem**

*It is relatively consistent with axioms of ZFC set theory that every real number, every function, every topological space, every set, is definable.*
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I shall give several proofs.
Motivating question

To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?
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To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?

This is what it would be like if the set-theoretic universe were pointwise definable.
Easy folklore observations

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If $\text{ZFC}$ is consistent, then there are continuum many non-isomorphic pointwise definable models of $\text{ZFC}$.

**Proof.**

Consider any $M \models \text{ZFC} + V = \text{HOD}$.
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Consider any $M \models ZFC + V = HOD$. Definable Skolem functions. Set of definable elements closed under the these Skolem functions, hence elementary, hence pointwise definable. So every completion of $ZFC + V = HOD$ has a pointwise definable model.
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By Gödel-Rosser, there are continuum many completions.
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*If ZFC is consistent, then there are continuum many non-isomorphic pointwise definable models of ZFC.*

**Proof.**

Consider any $M \models ZFC + V = \text{HOD}$. Definable Skolem functions. Set of definable elements closed under the these Skolem functions, hence elementary, hence pointwise definable.

So every completion of $ZFC + V = \text{HOD}$ has a pointwise definable model.

By Gödel-Rosser, there are continuum many completions.

Pointwise definable models with same theory are isomorphic. So these models are exactly all the pointwise definable models of ZFC.
Characterization of pointwise definability

That idea is fully general.

Observation

The following are equivalent:

1. $M$ is a pointwise definable model of $\text{ZFC}$.
2. $M$ consists of the definable elements of a model of $\text{ZFC} + V = \text{HOD}$.
3. $M$ is a prime model of $\text{ZFC} + V = \text{HOD}$.

Pointwise definability is a strong form of $V = \text{HOD}$.

We might introduce the notation $V = D$ or $V = \text{HD}$, but we don’t want to suggest that pointwise definability is first-order expressible.
Transitive pointwise definable models

Theorem

*If there is a transitive model of ZFC, then there are continuum many transitive pointwise-definable models of ZFC.*
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Fix transitive $N \models ZFC + V = \text{HOD}$. The definable elements of $N$ form an elementary substructure, whose Mostowski collapse is pointwise definable.
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For continuum many such models, force to add a Cohen real $N[c]$, and then force $V = \text{HOD}$ in $N[c][G]$ by coding into the GCH pattern, and make $c$ definable. The definable elements of $N[c][G]$ include $c$ and have pointwise definable Mostowski collapse. There is a perfect set of such $c$. 
## Minimal Transitive Model

**Theorem**

The minimal transitive model of \( \text{ZFC} \) is pointwise definable.
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This model is known as the Shepherdson-Cohen model—it is the smallest $L_\alpha$ that is a model of ZFC.
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The argument generalizes to show that the next-least ZFC-model $L_\beta$ after $L_\alpha$ is also pointwise definable, and indeed pointwise definability is pervasive in the countable $L$-hierarchy.
Pointwise definable ZFC extensions

The HOD-based arguments achieve pointwise definability by casting out the non-definable elements.
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Let me now explain how to achieve pointwise definability by adding new elements.
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Let me now explain how to achieve pointwise definability by adding new elements.

**Theorem**

*Every countable model of ZFC has a pointwise definable class forcing extension.*

Proved by myself, Linetsky, and Reitz in [HLR13]. Mentioned independently by Enayat in [Ena05].
Simpson’s Theorem

The proof uses a PA result of Simpson, applied to ZFC.

**Theorem (Simpson 1974)**

Let \( \langle M, \in \rangle \) be a countable model of ZFC. Then, there is an \( M \)-generic class \( U \subseteq M \) such that \( \langle M, \in, U \rangle \models ZFC(U) \) and every element of \( M \) is definable in \( \langle M, \in, U \rangle \).

**Proof.**

Use \( Q = \text{Add} (\text{ORD}, 1) \). Enumerate sets of ordinals of \( M \) as \( \langle a_n \mid n < \omega \rangle \). Enumerate dense classes \( \langle D_n \mid n < \omega \rangle \), where \( D_n \) is defined by \( \varphi_n (x, a_i)_{i < n} \). Define descending \( p_n \) so that \( p_{n+1} \) is the shortest extension of \( p_n \) in \( D_n \), followed by a block listing \( a_n \) and end-marker. Resulting filter \( U \subset \text{ORD} \) is \( M \)-generic, but every \( a_n \) is definable in \( \langle M, \in, U \rangle \).
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Start with countable model $\langle M, \in^M \rangle \models ZFC$. 
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- Conclusion: in $M[G]$, every set is definable without parameters.
New: pointwise definable end-extensions

Theorem (Hamkins [Ham22])

Every countable model of ZF has a pointwise definable end extension.
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Proof uses set-theoretic analogue of the universal algorithm.
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There is a definable finite sequence

$$a_0, a_1, \ldots, a_n$$

with the universal extension property for top-extensions.

If sequence is $s$ in countable $M \models ZFC$, then for any desired $t$, there is a top-extension $N \models ZFC$ in which the sequence is $t$. 
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Subtle, self-referential definition. “petulant child”
## Pointwise definable end-extensions

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Build a tower of progressively elementary extensions

$$M_0 \subseteq M_1 \prec_{\Sigma_1} M_2 \prec_{\Sigma_2} M_3 \prec_{\Sigma_3} \cdots$$
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Can arrange that every element becomes definable. So $N$ is pointwise definable.
Abundant pointwise definable models of set theory

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How do these models avoid the Math Tea argument?
Underlying the math tea argument is the presumption that we can associate every definition to the object it defines.

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It is by diagonalizing against this enumeration that one is supposed to conclude there are real numbers we cannot define.

But the fact is, we can’t always form this association in the first place.
The range of possibility

(i) There is no uniform definition of class of definable elements.

Specifically, there is no formula $df(x)$ in the language of set theory that is satisfied in any model $M \models \text{ZFC}$ exactly by the definable elements. To see this, consider $\forall x df(x)$ in a pointwise definable model and elementary extensions.
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For example, in a pointwise definable model.
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(ii) In some models, the class of definable elements is nevertheless definable.

For example, in a pointwise definable model.

(iii) In others, the definable elements do not form a class.

Consider any nontrivial ultrapower of a pointwise definable model.
More possibilities

(iv) The definable elements may be a class, but not $\psi_r \mapsto r$.

This is true in a pointwise definable model.
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(iv) The definable elements may be a class, but not $\psi_r \mapsto r$. This is true in a pointwise definable model.

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Diagonalize against it.
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Reveals subtle definability aspect to Frege/Russell interaction.
Summary Conclusion

Returning to the math-tea argument...  

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- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...
Thank you.

Joel David Hamkins
University of Notre Dame

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References

