The math tea argument—must there be numbers we cannot describe or define?

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The math tea argument

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> *"There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions."*

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> *"I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it."*
>
> *–Horatio, age 8*

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$$z = 0 \quad \iff \quad \langle \mathbb{Z}, + \rangle \models z + z = z.$$

No other elements are definable, because negation $x \mapsto -x$ is an automorphism.

Pointwise definability

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Thus, $\langle \mathbb{Z}, +, \cdot \rangle$ is *pointwise definable*: every individual is definable.

Ordered real field $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$

Note that the order $<$ is definable from algebraic structure

$$x < y \quad \iff \quad \exists a \neq 0 \quad x + a^2 = y.$$

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But only algebraic numbers are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.

Theorem (Tarski)

In the ordered real field $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$, every formula $\varphi(x)$ is equivalent to a quantifier-free formula.

One begins to see this by recalling

$$\exists x \; ax^2 + bx + c = 0 \quad \iff \quad b^2 - 4ac \geq 0.$$

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Corollary

The field of real algebraic numbers $\mathbb{A}$ is an elementary substructure of $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.

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There is a computable procedure to decide truth of any statement in $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$.

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The algebraic numbers are a real-closed field. So the elimination result works in the algebraic numbers. So $\mathbb{A}$ is an elementary substructure. So every definable element is algebraic.

Leibnizian models

A model M is *Leibnizian* if any two distinct points have different properties: if $a \neq b$, then there is some formula φ such that

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Are these notions the same?

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This is a successful instance of the Math Tea argument.

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Note that $\langle \mathbb{Z}, <, A \rangle$ is rigid, even though it has no definable elements.

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In particular, every computable real number and much more is definable.

Computable numbers

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Better idea: use rational approximations to z, to within a specified accuracy. Very robust conception.

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In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

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- or to define objects in $\langle V_\alpha, \in \rangle$ when α is not itself definable. (This amounts to using α as a parameter.)

We are thereby pushed:

- to allow only countable languages, and
- to consider only structures that are themselves definable with respect to the set-theoretic background $\langle V, \in \rangle$.

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Does it withstand scrutiny?

Well, it's complicated.

In a fixed structure

In a fixed set-sized structure $\mathcal{M}$ in a countable language and with all the real numbers in it, the math tea argument is fine: there are only countably many definitions, but uncountably many reals.

We simply associate each definable object r with a formula ψ_r that defines it. With access to such a definability map

$$\psi_r \mapsto r,$$

we may diagonalize against it to produce a real that is not definable.

Meta-mathematical obtacle

When defining reals r over the full set-theoretic universe $\langle V, \in \rangle$, however, a subtle meta-mathematical obstacle arises:

The property of *being definable in* $\langle V, \in \rangle$ is not first-order expressible in set theory.

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The key subtlety is that if we lack the association of definition with object defined, we cannot undertake the diagonalization to produce the non-definable real.

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By Tarski's theorem on the nondefinability of truth, this is impossible.

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We might expect that in any model of ZFC, there must be real numbers that are not definable in that model.

But that isn't true.

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I shall give several proofs.

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Motivating question

To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?

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To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?

This is what it would be like if the set-theoretic universe were pointwise definable.

Easy folklore observations

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Pointwise definable models with same theory are isomorphic. So these models are exactly all the pointwise definable models of ZFC.

Characterization of pointwise definability

That idea is fully general.

Observation

The following are equivalent:

1. M is a pointwise definable model of ZFC.
2. M consists of the definable elements of a model of $\mathrm{ZFC} + V = \mathrm{HOD}$.
3. M is a prime model of $\mathrm{ZFC} + V = \mathrm{HOD}$.

Pointwise definability is a strong form of $V = \mathrm{HOD}$.

We might introduce the notation $V = D$ or $V = HD$, but we don't want to suggest that pointwise definability is first-order expressible.

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Fix transitive $N \models \mathrm{ZFC} + V = \mathrm{HOD}$. The definable elements of N form an elementary substructure, whose Mostowski collapse is pointwise definable.

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For continuum many such models, force to add a Cohen real $N[c]$, and then force $V = \mathrm{HOD}$ in $N[c][G]$ by coding into the GCH pattern, and make c definable. The definable elements of $N[c][G]$ include c and have pointwise definable Mostowski collapse. There is a perfect set of such c. $\square$

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The argument generalizes to show that the next-least ZFC-model L_β after L_α is also pointwise definable, and indeed pointwise definability is pervasive in the countable L-hierarchy.

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Theorem

Every countable model of ZFC *has a pointwise definable class forcing extension.*

Proved by myself, Linetsky, and Reitz in [HLR13]. Mentioned independently by Enayat in [Ena05].

Simpson's Theorem

The proof uses a PA result of Simpson, applied to ZFC.

Theorem (Simpson 1974)

Let $\langle M, \in \rangle$ be a countable model of ZFC. Then, there is an M-generic class $U \subseteq M$ such that $\langle M, \in, U \rangle \models \mathrm{ZFC}(U)$ and every element of M is definable in $\langle M, \in, U \rangle$.

Proof.

Use $\mathbb{Q} = \mathrm{Add}(\mathrm{ORD}, 1)$. Enumerate sets of ordinals of M as $\langle a_n \mid n < \omega \rangle$. Enumerate dense classes $\langle D_n \mid n < \omega \rangle$, where D_n is defined by $\varphi_n(x, a_i)_{i<n}$. Define descending p_n so that p_{n+1} is the shortest extension of p_n in D_n, followed by a block listing a_n and end-marker. Resulting filter $U \subset \mathrm{ORD}$ is M-generic, but every a_n is definable in $\langle M, \in, U \rangle$. $\square$

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- Conclusion: in $M[G]$, every set is definable without parameters.

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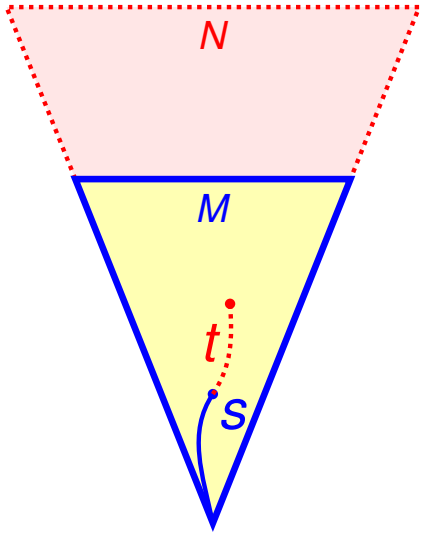
There is a definable finite sequence

$$a_0,\ a_1,\ \ldots,\ a_n$$

with the universal extension property for top-extensions.



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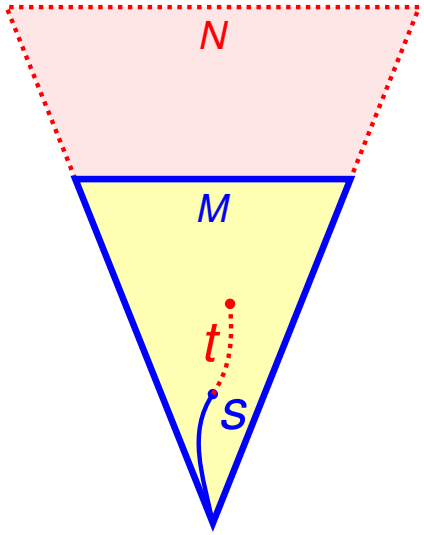

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Subtle, self-referential definition. “petulant child”

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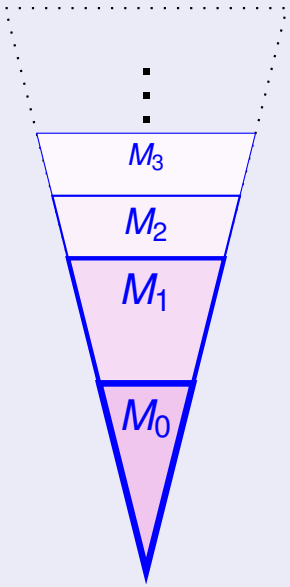

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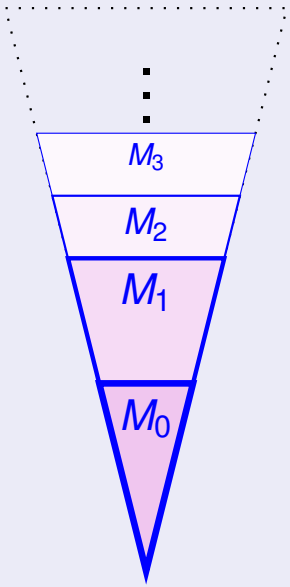

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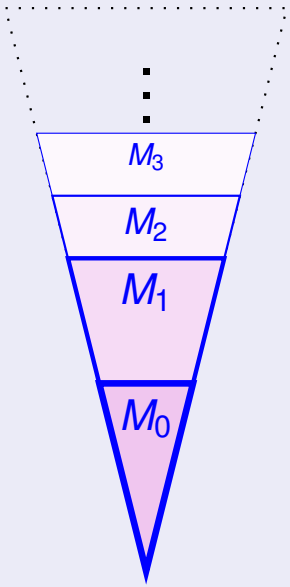

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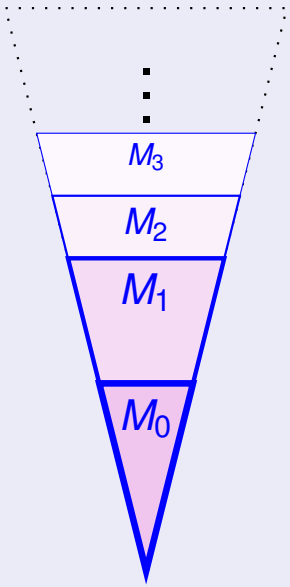

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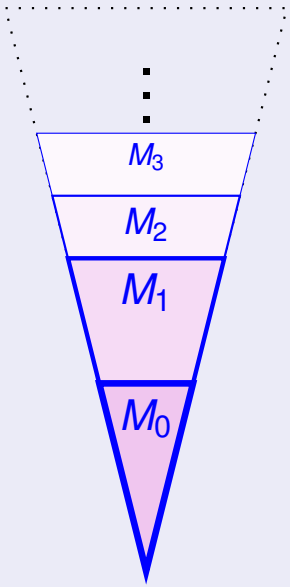

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Can arrange that every element becomes definable. So N is pointwise definable.

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Abundant pointwise definable models of set theory

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How do these models avoid the Math Tea argument?

Definability map

Underlying the math tea argument is the presumption that we can associate every definition to the object it defines.

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But the fact is, we can't always form this association in the first place.

The range of possibility

(i) There is no uniform definition of class of definable elements.

Specifically, there is no formula $\mathrm{df}(x)$ in the language of set theory that is satisfied in any model $M \models \mathrm{ZFC}$ exactly by the definable elements. To see this, consider $\forall x\, df(x)$ in a pointwise definable model and elementary extensions.

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(ii) In some models, the class of definable elements is nevertheless definable.

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(iii) In others, the definable elements do not form a class.

Consider any nontrivial ultrapower of a pointwise definable model.

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The surviving content of the math-tea argument: in any model with $\psi_r \mapsto r$, the definable reals do not exhaust all the reals.

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Reveals subtle definability aspect to Frege/Russell interaction.

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And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Thank you.

Joel David Hamkins
University of Notre Dame

http://jdh.hamkins.org

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