The surprising strength of second-order reflection in set theory with abundant urelements

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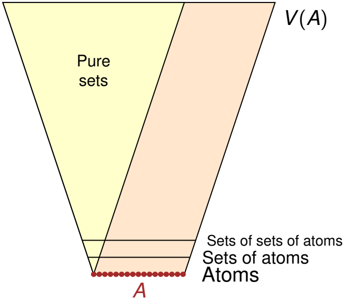
Joint work with Bokai Yao, PhD 2023 Notre Dame, now at Peking University.

- Joel David Hamkins and Bokai Yao. "Reflection in second-order set theory with abundant urelements bi-interprets a supercompact cardinal". *Journal of Symbolic Logic* (2023). to appear. arXiv:2204.09766[math.LO].
  http://jdh.hamkins.org/second-order-reflection-with-abundant-urelements
- Bokai Yao. "Reflection and Choice Principles with Absolute Generality". PhD thesis. University of Notre Dame, 2023
- Bokai Yao. "A hierarchy of principles in ZFC with urelements". under review. 2022

Urelements



Set theory was traditionally conceived as a theory of abstract collection over a class of already-existing primitive objects, the *urelements* or *atoms*.

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$V(A) = \bigcup_{w \subseteq A} V(w)$, where w ranges over sets.

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Atoms are set-theoretically indistinguishable—the universe is homogeneous with respect to atoms.

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Ultimately similar to the situation of ZFC^-. [GHJ16; Zar96]

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Main lesson

Axiomatization of urelement set theory can be finicky. Take care, since natural approaches are not the same.
[Yao23; Yao22]

Interpreting ZFCU in ZFC

Let me describe how to interpret various urelement set theories in ZFC.

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Bi-interpretation

Theorem

ZF *universe* $\langle V, \in\rangle$ *is bi-interpretable, for any class A, with* $\langle V[\![A]\!], \bar{\in}, \vec{\mathbb{A}}\rangle$, *a model of* $\mathrm{ZF}\vec{\mathrm{U}}$ + *"A many urelements."*

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Can interpret $V[\![A]\!]$ inside V.

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Need enumeration predicate $\vec{\mathbb{A}}$ and not just $\mathbb{A}$. □

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Similar to corresponding two views of ultrapowers $j : V \to M$ and also forcing $V \subseteq V[G]$.

Bi-interpretable theories

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6. $\text{ZFC}\vec{\text{U}}$ + "there are *V* many atoms"

Bi-interpretation works in ZFCU with parameters, if *A* is a set.

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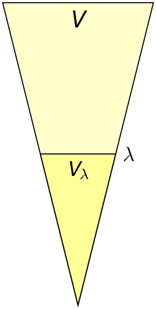
Conclusion: its totally different to have $\mathbb{A}$ versus $\vec{\mathbb{A}}$.

Reflection in ZF



Lévy-Montague reflection theorem

For any $\varphi(x)$, there is λ with φ absolute between V_λ and V.

Indeed, for every k there is a club of Σ_k-correct cardinals λ.

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Conceivable that very bad sets of urelements could spoil reflection.

Class theory with pure sets

Models have form $\langle M, \in^M, \mathcal{M}\rangle$, where $\langle M, \in^M\rangle$ is a model of ZFC and $\mathcal{M} \subseteq P(M)$ is collection of classes $X \subseteq M$.

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Even KM does not prove class-choice CC principle ([GH17]).
(but $\mathrm{KM}+\mathrm{CC}$ mutually interpretable with KM)

Interpretation of Kelley-Morse in a first-order set theory

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The following theories are bi-interpretable.

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Use CC to achieve collection axiom. □

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Warning: this is strictly stronger than global choice function.

Interpretations in class urelement theories

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If the original model satisfies KM, then the interpreted model satisfies KMU, and if the original model satisfies the class choice principle CC, then so does the interpreted model.

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Bokai Yao was particularly motivated by the possibility of having a huge class of atoms.

Motivating case

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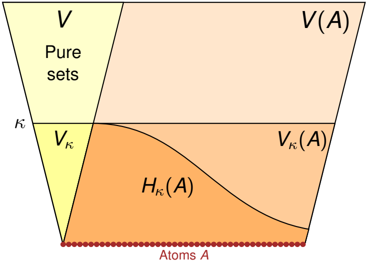

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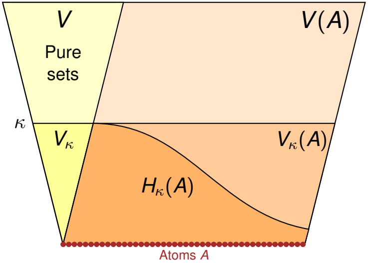

Consider $\langle H_\kappa(A), \in, \mathcal{H}\rangle$, where $\mathcal{H} = \mathcal{V} \upharpoonright H_\kappa(A)$.

A can be enormous relative to κ, but still a class in $H_\kappa(A)$.

Abundant atom axiom

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Define class B is *small*, if size less than A.

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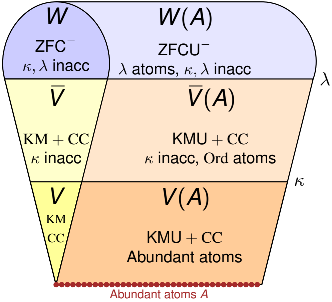
AAA is true in $\langle H_\kappa(A), \in, \mathcal{H}\rangle$.

Bi-interpretation Theorem. The following theories are bi-interpretable

1. KMU $+\,\mathrm{CC}+$ the abundant atom axiom
2. $\mathrm{KM}+\mathrm{CC}+\exists$ inaccessible
3. KMU $+\,\mathrm{CC}+\exists$ inaccessible $+\,\mathrm{Ord}$ many atoms
4. $\mathrm{ZFC}^- + \exists\kappa<\lambda$ inaccessible, λ largest cardinal
5. $\mathrm{ZFCU}^- + \exists\kappa<\lambda$ inaccessible, λ largest, λ atoms



Second-order reflection

Second-order reflection principle

Every second-order $\varphi(X)$ true in $\langle V, \in, \mathcal{V}\rangle$ is true in some $V_\lambda \models \varphi(X \cap V_\lambda)$.

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Get a stationary class of Π^1_n-indescribable cardinals...

A curiosity

Reflection erases the difference between GBc and KM+CC.

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Theorem

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Theorem

Assume GBc plus second-order reflection. Then KM holds, including global choice, plus CC.

Proof.

If second-order comprehension failed, this would reflect to some inaccessible V_κ, but KM holds there.

Similarly, if there were no global well-order, this would reflect to V_κ, but by AC it holds there. □

Upper bound on second-order reflection

Theorem

If κ is a measurable cardinal, then $\langle V_\kappa, \in, V_{\kappa+1}\rangle$ is a model of $\mathrm{KM} + \mathrm{CC}$ plus the second-order reflection principle.

Proof.

Suppose $\varphi(X)$ true in $\langle V_\kappa, \in, V_{\kappa+1}\rangle$. Consider $j : V \to M$ with critical point κ.

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ω-Erdős suffices.

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If κ is κ^+-supercompact, then there is a model of KMU with more than Ord many urelements in which second-order reflection holds.

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Was supercompactness really required?

Second-order reflective cardinals

Definition ([HY23])

1. κ is *second-order reflective*, if every second-order φ true in some structure *M* language size $< \kappa$ with $\kappa \subseteq M$ is true in some $m \prec M$ size $< \kappa$ with $m \cap \kappa \in \kappa$.

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Natural affinity with Magidor's characterization [Mag71] of least supercompact. Nearly the same, except for $m \cap \kappa \in \kappa$.

Shades of supercompactness

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Proof.

Consider any $M = \langle \lambda, R, \cdots \rangle \models \varphi$, and let $j : V \to N$ be a λ-supercompact embedding. Note $j(M) = \langle j(\lambda), j(R), \cdots \rangle$ reflects to $j \text{ " } M = \langle j \text{ " } \lambda, j(R), \cdots \rangle$, which is isomorphic to M. □

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If κ is $2^{\lambda^{<\kappa}}$-reflective for Π^1_1 assertions, then κ is λ-supercompact.

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Theorem

If κ is $2^{\lambda^{<\kappa}}$-reflective for Π^1_1 assertions, then κ is λ-supercompact.

Proof.

Let $M = \langle H_{(\lambda^{<\kappa})^+}, \in, \kappa, \lambda \rangle$. Nonexistence of normal fine measure on $P_\kappa \lambda$ is second-order Π^1_1-expressible. But over set $m \prec M$ can build normal, fine m-measure using seed $s = m \cap \lambda$. □

Characterization of supercompact

Corollary ([HY23])

The following are equivalent:

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Constellation of closely related results: [Car85, theorems 3.5, 4.7], [Cod20, theorem 1.4], [HM22, lemma 2.8].

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Consider second-order reflection in the urelement context.

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If κ is λ-supercompact, $\lambda > \kappa$, then in $V[\![\lambda]\!]$ the model $\langle H_\kappa(A), \in, \mathcal{H}\rangle$ is a model of KMU $+ \mathrm{CC}$ *plus second-order reflection with more than* Ord *many urelements, indeed,* λ *many.*

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Proof.

$H_\kappa(A)$ has size λ, so this flows from κ being λ-reflective. □

This generalizes and explains Yao's original construction.

Improved result

Weaken λ-supercompactness to nearly λ-supercompact.

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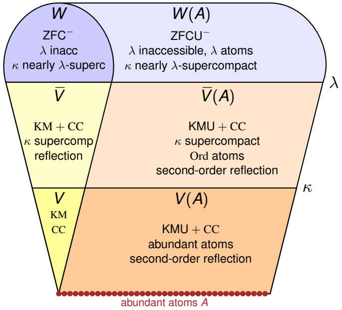
If κ is nearly λ-supercompact, then in $V[\![\lambda]\!]$ the model $\langle H_\kappa(A), \in, \mathcal{H}\rangle$ satisfies KMU $+$ CC plus Π^1_1-reflection with more than Ord many urelements, indeed, λ many.

Our main theorem, next, shows that full supercompactness is required when there are abundant urelements.

Main Theorem (Hamkins-Yao [HY23]). Following theories are bi-interpretable.

0. GBUc + abundant atom axiom + second-order reflection
1. KMU + CC + abundant atom axiom + second-order reflection
2. $\mathrm{KM} + \mathrm{CC} + \kappa$ supercompact, Ord-reflective
3. KMU + $\mathrm{CC} + \mathrm{Ord}$ atoms + κ supercompact, Ord-reflective
4. ZFC$^-$ + κ is $<\lambda$-supercompact, λ-reflective, λ largest, inaccessible
5. ZFCU$^-$ + λ atoms + κ is $<\lambda$-sc, λ-reflective, λ largest, inaccessible



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Main lesson of the main theorem.

- If you want more than Ord many urelements with classes, you didn't grow the universe tall enough
- But again, not if one wants to allow weird sets of atoms

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
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Oxford University

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