The Contingent HOD Dichotomy

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Introduction

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I shall identify and prove a certain fundamental dichotomy, which we call the contingent HOD dichotomy. Namely,

1. Either the universe is close to HOD in several respects, and this is preserved by forcing throughout the generic multiverse;
2. Or the universe is far from HOD in those respects, and this also is preserved by forcing throughout the generic multiverse.

Review of ordinal definability

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I should like to emphasize that, on its face, this definition is an external, model-theoretic notion.

The class OD

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In light of this, we move to the transitive inner core of ordinal definability. Namely, we define HOD to consist of the *hereditarily* ordinal definable sets a, meaning that a is ordinal definable along with all its elements, elements of elements—the set a and all its hereditary elements are ordinal definable.

$$a \in \mathrm{HOD} \quad \iff \quad \mathrm{TC}(\{a\}) \subseteq \mathrm{OD}.$$

HOD is a transitive inner model of set theory.

A philosophical puzzle about ordinal definability

There is a conundrum at the heart of ordinal definability.

Question

How is it possible in the first place to define ordinal definability in the language of set theory?

Have we actually defined OD and HOD as classes in ZFC?

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Metatheory versus object theory. It was an external, model-theoretic definition—we quantified over formulas φ and referred to truth of $\varphi(a, \vec{\alpha})$.

But truth is not definable.

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The property "x is definable" is not generally expressible in set theory.

Proof: Consider "the least undefinable ordinal." This amounts to a transfinite version of Berry's paradox.

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Pointwise definable models [HLR13; Ham24].

Interesting case: ω-standard models with a correct cardinal, $V_\kappa \prec V$. Definability is definable from κ.

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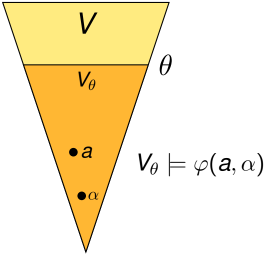
How are we able to define “x is ordinal definable” when we cannot express “x is definable”?

Ordinal definability is definable

A standard approach: define that a is *ordinal-definable* if a is definable in some $\langle V_\theta, \in \rangle$ from ordinal parameters.



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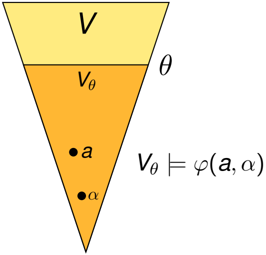

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This is expressible in ZFC since we have a satisfaction relation for the structure $\langle V_\theta, \in\rangle$.

In fact, this definition aligns with the metatheoretic notion.

The easy direction, via reflection

If a is actually definable in V from ordinal parameters via $\varphi(a, \vec{\alpha})$, then by reflection this will also hold in some V_θ.

So every object that is ordinal definable in V is also ordinal definable in some V_θ and hence in OD.

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We are defining a in V using parameters $(\theta, \vec{\alpha})$.

Is it right?

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> *The thing for which $\varphi(a, \vec{\alpha})$ holds, relativized to V_θ.*

We are defining a in V using parameters $(\theta, \vec{\alpha})$.

Is it right?

Unfortunately, not quite. We sometimes need different/additional parameters.

Subtle problems

Suppose *a* is definable in V_θ, so $\exists\varphi, \vec{\alpha}$ s.t. *a* defined by $\varphi(a, \vec{\alpha})$.

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Worse, the arity might be nonstandard. The tuple $\vec{\alpha}$ could be a nonstandard finite sequence. Infinitely many parameters.

Makes no sense to use $\varphi(x, \vec{\alpha})$ in the metatheory.

Consider $M \models \mathrm{ZFC}$ with uncountable $\mathbb{N}^M$. It thinks every element of V_ω^M is definable without parameters, but there are too many.

Solution to nonstandard tuples of ordinals

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Every pair α, β is definable in some V_θ.

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Proof 2. Take least $\vec{\alpha}$ not definable in any V_θ, then reflect. Contradiction.

By induction, can define arbitrary $\vec{\alpha}$, including nonstandard tuples.

So x is ordinal definable if and only if it is definable in some $\langle V_\theta, \in \rangle$ without any parameters.

In effect, θ itself serves as the only ordinal parameter needed.

Solution to nonstandard φ

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Solution to nonstandard φ

Suppose *a* is definable via φ in $\langle V_\theta, \in\rangle$, but perhaps φ is nonstandard.

Then we can define *a* in *V* as:

> *the unique object for which the formula coded by* $\ulcorner\varphi\urcorner$ *is true of a in* V_θ*.*

This uses parameter $\ulcorner\varphi\urcorner$, which is an ordinal. Truth is definable in $V_{\theta+1}$.

So *a* is ordinal definable in the metatheoretic sense.

Conclusion: the metatheoretic notion of ordinal definable is expressible in the language of set theory.

Alternative resolution

In V we can define a global well-order of OD.

$a \leq_{\mathrm{OD}} b \iff$ a is definable in smaller V_θ than b, or in same V_θ but with a smaller φ, or with same least V_θ and φ, but with smaller parameter (or equal).

This is how we know $\mathrm{HOD} \models \mathrm{AC}$.

But also: every ordinal definable a is the αth element with respect to $\leq_{\mathrm{OD}}$.

A second puzzle, regarding history

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> *Here also you have, corresponding to the transfinite hierarchy of formal systems, a transfinite hierarchy of concepts of definability... The simplest way of doing it is to take the ordinals themselves as primitive terms. So one is led to the concept of definability in terms of ordinals, — i.e., definability by expressions containing names of ordinal numbers and logical constants, including quantification referring to sets. This concept should, I think, be investigated. It can be proved that it has the required closure property: By introducing the notion of truth for this whole transfinite language, i.e., by going over to the next language, you will obtain no new definable sets (although you will obtain new definable properties of sets). [Gö90, p.151]*

Gödel does not elaborate with much technical detail, but clearly proposes to express the concept of ordinal definability in terms of a formal truth predicate.

Use of reflection

Gödel mentions that "by going over to the next language, you will obtain no new definable sets."

We can prove this by reflection in expanded language. If a is definable in language with Tr, then let θ be $\Sigma_1(\mathrm{Tr})$-correct above a. So $\mathrm{Tr} \restriction V_\theta$ fulfills Tarski recursion. So Tr agrees with satisfaction in $\langle V_\theta, \in\rangle$. In particular, $V_\theta \prec V$. So a is definable in V_θ without Tr, since we just refer to satisfaction in V_θ instead.

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Using the reflection theorem, we see that Gödel was completely correct.

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How did Gödel do it? Did Gödel know the reflection theorem? Did he have an alternative proof in mind?

Did Gödel have the reflection theorem?

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In our view, however, the evidence is thin.

In any case, Gödel's remarks are brief, and lack technical detail.

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Resolving the historical paradox

Q: How did Gödel prove that ordinal definability is definable without reflection?

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Resolving the historical paradox

Q: How did Gödel prove that ordinal definability is definable without reflection?

A: He did not prove or even claim this.

Contingent contingency of $V = \mathrm{HOD}$

The axiom $V = \mathrm{HOD}$ is contingently contingent by forcing.

In some models of set theory, $V = \mathrm{HOD}$ is necessarily *contingent*—you can turn it on and off by forcing. It is a *switch*.

$$\Box(\Diamond(V = \mathrm{HOD}) \wedge \Diamond(V \neq \mathrm{HOD})).$$

In other models, $V = \mathrm{HOD}$ is necessarily not contingent, in fact, necessarily false by forcing.

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In fact, these are the only two possibilities.

Forcing $V \neq \mathrm{HOD}$

Observation (well-known)

$V \neq \mathrm{HOD}$ is easily forceable over any model of ZFC.

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Proof.

In fact, $V \neq \mathrm{HOD}$ after forcing to add a Cohen real.

More generally, $\mathbb{P}$ is *homogeneous* if $\forall p, q \in \mathbb{P}$ there are $p' \leq p$ and $q' \leq q$ such that $\mathbb{P} \upharpoonright p'$ is forcing-equivalent to $\mathbb{P} \upharpoonright q'$.

If $\mathbb{P}$ is homogeneous, then all conditions force the same statements $\varphi(\check{a})$ about ground model objects, including ordinals.

It follows that $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}(\mathbb{P})^V$. In particular, $\mathrm{HOD}^{V[G]} \subseteq V$.

So homogeneous forcing forces $V \neq \mathrm{HOD}$ in $V[G]$. $\square$

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Common to use bookkeeping argument, carefully arranged to make sure every set (old and newly added) is coded.

But, one can dispense with the bookkeeping.

Use full-support Ord-iteration $\mathbb{P}$, which at regular δ forces with the *lottery sum* of forcing of GCH at δ or negation.

The generic filter $G \subseteq \mathbb{P}$ decides generically whether GCH holds at δ or not, by opting in each lottery. Generic bookkeeping.

This forces CCA—every set is coded into the GCH pattern.

New sparse method of forcing ordinal definability

Theorem

Assume SCH. *Then there is class forcing of $V = \mathrm{HOD}_b$, preserving cardinals, cofinalities, and continuum function 2^κ.*

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Assume SCH*. Then there is class forcing of* $V = \mathrm{HOD}_b$*, preserving cardinals, cofinalities, and continuum function* 2^κ*.*

Proof.

Let $\mathbb{P}$ be Easton-support iteration: add Cohen real, then at singular strong limit δ, use lottery sum of $\mathrm{Add}(\delta^+, 1)$ and trivial forcing.

Consider extension $V[G]$. Factor arguments show cardinals, cofinalities are preserved. Also 2^κ.

Set-theoretic geology shows V is definable from parameter $b = \mathbb{R}^V$ in $V[G]$.

Key point: $V[G]$ can tell which stages δ opted for trivial forcing.

Every set is coded in that pattern, so $V[G] = \mathrm{HOD}_{\mathbb{R}^V}$ in $V[G]$. $\square$

Class forcing versus set forcing

We forced $V = \mathrm{HOD}$ with class forcing.

Is it possible ever to force $V = \mathrm{HOD}$ with set-sized forcing?

For example, after adding a Cohen real $V[G][c]$, then $V \neq \mathrm{HOD}$. Can we recover $V = \mathrm{HOD}$ with further set forcing?

Answer:

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For example, after adding a Cohen real $V[G][c]$, then $V \neq \mathrm{HOD}$. Can we recover $V = \mathrm{HOD}$ with further set forcing?

Answer: First, you can never force CCA by nontrivial set forcing.

Second, nevertheless, you can always recover $V = \mathrm{HOD}$ by set forcing, if it holds in a ground.

Indeed, the axiom $V = \mathrm{HOD}$, when true, is a switch. You can turn it on and off by set forcing, starting from any model where it is true.

$V = \mathrm{HOD}$ can be impossible by set forcing

Theorem

In some models of ZFC*, the axiom $V = \mathrm{HOD}$ is not forceable.*

$V = \mathrm{HOD}$ can be impossible by set forcing

Theorem

In some models of ZFC, the axiom $V = \mathrm{HOD}$ is not forceable.

Proof.

Start in $V \models \mathrm{ZFC}$. Let $\mathbb{P} = \prod_{\delta \text{ reg}} \mathrm{Add}(\delta, 1)^V$, Easton support product.

Homogeneous forcing, so $V \neq \mathrm{HOD}$ in $V[G]$.

Consider any set forcing extension $V[G][h]$, with $h \subseteq \mathbb{Q} \in V[G]$.

Choose $\delta \gg \mathbb{Q}$ and let $g_\delta \subseteq \mathrm{Add}(\delta, 1)$ be generic added by G at δ.

Rearrange the forcing as

$$V[G][h] = V[G^*_\delta][h][g_\delta],$$

where G^*_δ omits coordinate δ.

Last step is homogeneous forcing, so $V \neq \mathrm{HOD}$ in $V[G][h]$. $\square$

When true, $V = \mathrm{HOD}$ is a switch

Theorem (Hamkins [Ham15])

$V = \mathrm{HOD}$, if true, is forceable over every forcing extension.

$$V = \mathrm{HOD} \quad\Longrightarrow\quad \Box\Diamond(V = \mathrm{HOD}).$$

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Proof.

Assume $V = \mathrm{HOD}$, and consider $V[g]$, generic $g \subseteq \mathbb{P} \in V$.

Perform *self-encoding forcing* above $\mathbb{P}$. That is, $V[g][h]$, where h codes g and big V_θ and h itself into GCH pattern.

Ground model V is definable from parameter $P(\mathbb{P})^V$ by approximation/cover. But that is definable from ordinals. So $V \subseteq \mathrm{HOD}^{V[g][h]}$. But g and h are OD, so $V[g][h] = \mathrm{HOD}^{V[g][h]}$.

So $V = \mathrm{HOD}$, when true, is a switch. $\square$

Equivalencies of V close to HOD

Theorem. The following are equivalent

1. The inner model HOD is a ground, $V = \mathrm{HOD}[G]$
2. The axiom $V = \mathrm{HOD}$ is forceable. $\Diamond(V = \mathrm{HOD})$
3. The universe is ordinal-definable from a set. $\exists b\ V = \mathrm{HOD}_b$
4. There is a definable global well order (w. parameters).
5. $V = \mathrm{HOD}$ holds somewhere in the generic multiverse.
6. HOD^W is a ground of W throughout generic multiverse.
7. $V = \mathrm{HOD}$ is forceable over every W in generic multiverse.
8. Every model W in the generic multiverse is ordinal-definable from a set. $\exists b\ W = \mathrm{HOD}_b^W$
9. Every W in generic multiverse has definable global well order (w. parameters).

Some ideas from the proof

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If HOD is a ground, so $V = \mathrm{HOD}[G]$, then $V = \mathrm{HOD}_G$.

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If HOD is a ground, so $V = \mathrm{HOD}[G]$, then $V = \mathrm{HOD}_G$.

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If $V = \mathrm{HOD}_b$, then Vopěnka's theorem shows b is generic over HOD and in fact HOD is a ground.

$V = \mathrm{HOD}_b$ is equivalent to b-definable global well order.

If $V = \mathrm{HOD}$ is forceable, then $V = \mathrm{HOD}_b$, where b codes the forcing notion and all conditions.

The contingent HOD dichotomy

Corollary

ZFC *proves the following dichotomy. Either*

1. *The following statements hold in V and throughout the generic multiverse.*
   1. *The inner model* HOD *is a ground.*
   2. $\Diamond(V = \mathrm{HOD})$.
   3. *The universe is ordinal-definable from a set.*
   4. *There is a definable global well order (from parameters).*
2. *Or those statements fail in V & throughout generic multiverse.*
   1. *The inner model* HOD *is not a ground.*
   2. $\Box(V \neq \mathrm{HOD})$.
   3. *The universe is not ordinal-definable from a set.*
   4. *There is no definable global well order (from parameters).*

If ZFC *is consistent, then both horns of this dichotomy occur.*

Contingent HOD dichotomy in canonical models

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Theorem

If $V = L[A]$, then first horn $\Diamond(V = \mathrm{HOD})$.

This covers L, L_μ, many further instances $L[\vec{E}]$. Whenever $V = \mathrm{HOD}$, then of course in first horn.

Ultrapower axiom

Theorem ([Gol22, Theorem 6.2.8])

Assume the Ultrapower Axiom and that there is a supercompact cardinal. Then HOD *is a ground of* V*.*

So these models are in the first horn of the contingent HOD dichotomy.

Contingent HOD dichotomy and the GCH

Theorem

The GCH is relatively consistent with either horn of the contingent HOD dichotomy.

Contingent HOD dichotomy and the GCH

Theorem

The GCH is relatively consistent with either horn of the contingent HOD dichotomy.

Proof.

Our earlier model of $\Box(V \neq \mathrm{HOD})$ has GCH.

But also $V = \mathrm{HOD} + \mathrm{GCH}$ is true in many canonical models. $\square$

Contingent HOD dichotomy and Easton's theorem

Theorem

Assume ZFC $+$ GCH *and suppose that E is an Easton function. Then there are cardinal-preserving class-forcing extensions $V[G]$ and $V[H]$, both satisfying* ZFC *and also $2^\kappa = E(\kappa)$ for every regular cardinal, such that $V[G]$ is in one horn of the contingent* HOD *dichotomy and $V[H]$ is in the other.*

Contingent HOD dichotomy and Easton's theorem

Theorem

Assume ZFC $+$ GCH *and suppose that* E *is an Easton function. Then there are cardinal-preserving class-forcing extensions* $V[G]$ *and* $V[H]$*, both satisfying* ZFC *and also* $2^\kappa = E(\kappa)$ *for every regular cardinal, such that* $V[G]$ *is in one horn of the contingent* HOD *dichotomy and* $V[H]$ *is in the other.*

The standard Easton-product proof provides $\Box(V \neq \mathrm{HOD})$.

Meanwhile, doing it as an Easton-support iteration forces also $V = \mathrm{HOD}_b$.

Q: Can we force $V = \mathrm{HOD}$ and achieve Easton's theorem simultaneously?

The contingent HOD dichotomy and large cardinals

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All the usual large cardinal hypotheses are consistent with both horns of the contingent HOD *dichotomy.*

Well-known how to force $V = \mathrm{HOD}$ via class forcing while preserving large cardinals.

Meanwhile, all the standard large cardinals are also preserved by the forcing of $\square(V \neq \mathrm{HOD})$.

Contingent HOD dichotomy and bedrocks

Theorem

If the set-theoretic universe V admits a bedrock, then we are in the $\Diamond(V = \mathrm{HOD})$ horn of the contingent HOD dichotomy if and only if the bedrock is a model of $V = \mathrm{HOD}$.

Contingent HOD dichotomy and bedrocks

Theorem

If the set-theoretic universe V admits a bedrock, then we are in the $\Diamond(V = \mathrm{HOD})$ horn of the contingent HOD dichotomy if and only if the bedrock is a model of $V = \mathrm{HOD}$.

Proof.

If the bedrock satisfies $V = \mathrm{HOD}$, then we are in the $\Diamond(V = \mathrm{HOD})$ case of the dichotomy.

Alternatively, if $V \neq \mathrm{HOD}$ in the bedrock W, then HOD^W is not a ground of W, since the only ground of a bedrock is the bedrock itself, so we are in the other horn. □

Contingent HOD dichotomy does not imply bedrock

Theorem

If ZFC is consistent, then both horns of the contingent HOD dichotomy are consistent both with the existence and with the nonexistence of a bedrock. All four combinations occur.

Proof.

We have $V = \mathrm{HOD}$ and a bedrock in L and other canonical models.

Main result of [HRW08] has ground axiom plus $V \neq \mathrm{HOD}$.

Bottomless model of Reitz is achieved with homogeneous class product, so $\Box(V \neq \mathrm{HOD})$.

Remaining case is bottomless + $V = \mathrm{HOD}$. Use class product of nonoverlapping self-encoding forcing over a model of CCA.

$\square$

Contingent HOD dichotomy and geology

Theorem

In the $\Diamond(V = \mathrm{HOD})$ horn of the contingent HOD dichotomy, the mantle is equal to the generic HOD.

$$\mathbb{M} = \mathrm{gHOD}$$

The mantle $\mathbb{M}$ is the intersection of all grounds, while gHOD is in the intersection of all $\mathrm{HOD}^{V[g]}$.

Key observation: In first horn, if $g \subseteq \mathbb{P}$ is homogeneous, then $\mathrm{HOD}^{V[g]} \subseteq V$, hence is a ground of V.

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Key observation: In first horn, if $g \subseteq \mathbb{P}$ is homogeneous, then $\mathrm{HOD}^{V[g]} \subseteq V$, hence is a ground of V.

Meanwhile, $\mathbb{M} = \mathrm{gHOD}$ is not necessarily equivalent to $\Diamond(V = \mathrm{HOD})$, since (genuinely) proper class homogeneous forcing always places us in the second horn.

The contingent HOD dichotomy and iterated HODs

Assume first horn $\Diamond(V = \mathrm{HOD})$ of the dichotomy.

So HOD is a ground. Also $\mathrm{HOD}^{\mathrm{HOD}}$, and $\mathrm{HOD}^{\mathrm{HOD}^{\mathrm{HOD}}}$.

All HOD^n are grounds.

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Subtle issues. Even HOD^ω is not generally definable in ZFC.

But in first horn, it is definable.

Question

In the first horn, are all $\mathrm{HOD}^{\alpha+1}$ grounds?

Iterated HODs

Question

Assume $\Diamond(V = \mathrm{HOD})$ and there is a bedrock. Then, does every W have it's HOD^α sequence eventually stabilizing?

The bedrock is a lower bound for the iterated HOD^α. Does this cause them to stabilize?

If they are all grounds, then they would have to stabilize, since when there is a bedrock, there are only a set number of grounds.

When there is a bedrock, the generic multiverse is set-like.

$V = \mathrm{HOD}$ in ground models?

Open Question

Is it consistent with ZFC that $V \neq \mathrm{HOD}$ holds in all ground models, yet $V = \mathrm{HOD}$ holds in some forcing extension?

Such a model must be bottomless, no bedrock. Every ground W would sit atop a strictly descending chain of $(\mathrm{HOD}^n)^W$.

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Is it consistent with ZFC that $V \neq \mathrm{HOD}$ holds in all ground models, yet $V = \mathrm{HOD}$ holds in some forcing extension?

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More positive framing:

Open Question

Does $\Diamond(V = \mathrm{HOD})$ imply there is a ground model with $V = \mathrm{HOD}$?

Maximality Principle

The *maximality principle* MP is the scheme of assertions that every possibly necessary statement is already true.

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$$\Diamond\Box\varphi \implies \varphi.$$

The boldface principle $\mathrm{MP}(\mathbb{R})$ allows real parameters in φ.

Uncountable parameters are impossible.

Introduced by Stavi and Väänänen [SV02], independently by Hamkins [Ham03], with the forcing modalities.

Contingent HOD dichotomy and MP

Theorem

The contingent HOD *dichotomy is independent of* ZFC $+ \mathrm{MP}$*. Indeed, the following theories are equiconsistent:*

1. ZFC $+ \mathrm{MP} + (V = \mathrm{HOD})$
2. ZFC $+ \mathrm{MP} + \Box(V \neq \mathrm{HOD})$
3. ZFC $+ \mathrm{MP}$
4. ZFC
5. ZFC $+ (V_\delta \prec V)$
6. ZFC $+ (V_\delta \prec V) + (V = \mathrm{HOD})$
7. ZFC $+ (V_\delta \prec V) + \Box(V \neq \mathrm{HOD})$

Contingent HOD dichotomy and $\mathrm{MP}(\mathbb{R})$

Theorem

The contingent HOD *dichotomy is independent of* ZFC $+ \mathrm{MP}(\mathbb{R})$. *More specifically, the following theories are equiconsistent:*

1. ZFC $+ \mathrm{MP}(\mathbb{R}) + (V = \mathrm{HOD})$.
2. ZFC $+ \mathrm{MP}(\mathbb{R}) + \Box(V \neq \mathrm{HOD})$.
3. ZFC $+ \mathrm{MP}(\mathbb{R})$.
4. ZFC $+ \mathrm{Ord}$ *is definably Mahlo.*
5. ZFC $+ (V_\delta \prec V) + (\delta$ *is inaccessible*$)$.
6. ZFC $+ (V_\delta \prec V) + (\delta$ *is inaccessible*$) + (V = \mathrm{HOD})$.
7. ZFC $+ (V_\delta \prec V) + (\delta$ *is inaccessible*$) + \Box(V \neq \mathrm{HOD})$.

The other HOD dichotomy

Theorem (Woodin's HOD dichotomy [WDR12])

Suppose that δ is an extendible cardinal. Then exactly one of the following alternatives holds.

1. *Every singular cardinal $\lambda > \delta$ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$.*
2. *Every regular cardinal $\kappa > \delta$ is ω-strongly measurable in HOD.*

Both dichotomies fall under the slogan, "V is close to HOD" versus "V is far from HOD."

But they are not equivalent, and indeed, Woodin conjectured that his second horn does not occur (the HOD conjecture).

Our dichotomy splits first horn of Woodin's dichotomy

Our first horn $\Diamond(V = \mathrm{HOD})$ is entirely contained within Woodin's main horn.

Proof: if HOD is a ground, then above the forcing, it is correct about all regular cardinals. So they will not all be measurable in HOD.

Our dichotomy splits first horn of Woodin's dichotomy

Our first horn $\Diamond(V = \mathrm{HOD})$ is entirely contained within Woodin's main horn.

Proof: if HOD is a ground, then above the forcing, it is correct about all regular cardinals. So they will not all be measurable in HOD.

Meanwhile, we can force our second horn $\Box(V \neq \mathrm{HOD})$ by class forcing that preserves Woodin's first horn.

So the dichotomies are not identical.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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