The covering reflection principle

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This is joint work in progress with myself, Nai Chung Hou, Andreas Leitz, and Farmer Schlutzenberg.

The topic originated in Hou's question [Hou23] on MathOverflow and our various answers to it [Ham23; Sch23; Lie23], in which the solution emerged gradually, ultimately converging to the current collaboration [Ham+].

The covering reflection principle

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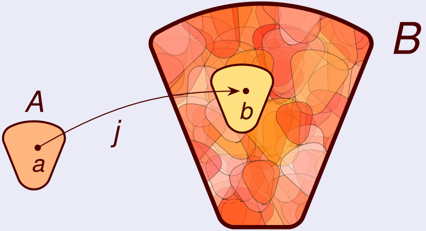
. . . but it has a set-theoretic core."

The covering reflection principle

Covering reflection principle CRP_δ

Holds for a cardinal δ, if for every first-order structure B in a countable language, there is substructure A, size less than δ, such that B is covered by the elementary images of A in B.



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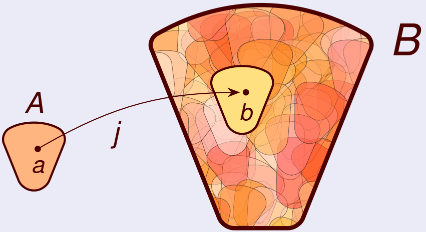

That is, every element $b \in B$ is in the range of some elementary embedding $j : A \to B$.

Instances of covering reflection

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Instances of covering reflection

Model theory is full of instances of covering reflection.

- Every uncountable dense linear order, for example, is covered by elementary images of a fixed countable dense linear order.
- Similarly, every model of an $\aleph_0$-categorical theory is covered by elementary images of the unique countable model.
- In fact, same is true for κ-categorical theories in uncountable powers κ—they are covered by elementary images of a fixed countable structure.

Models of κ-categorical theories

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Furthermore, it is strongly uniform—there is a countable $A \models T$ covering every uncountable $B \models T$ by its elementary images.

Proof.

$\aleph_0$-categorical is easy case—cover by countable elementary substructures.

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Proof.

$\aleph_0$-categorical is easy case—cover by countable elementary substructures.

κ-categorical for uncountable κ. By Morley, all uncountable $B \models T$ are saturated. Morley also proved T is $\aleph_0$-stable, so there is a countable saturated model. It covers. □

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Question

Is there any such δ? Does covering reflection occur? How large is the smallest cardinal exhibiting covering reflection? Is the covering reflection principle consistent? What is the consistency strength?

Easy observations

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Closed upward

If covering reflection holds for δ, then also for any larger $\delta' > \delta$.

So our focus might be placed on the smallest δ for which covering reflection holds.

Must be uncountable

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Covering reflection fails for $\delta = \aleph_0$, since the small model A would have to be finite, but no infinite model B has finite elementary substructures.

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So δ must be uncountable. $\omega_1 \leq \delta$.

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How big must δ be? Is there any δ at all with covering reflection?

Natural variations are equivalent

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Proof.

Given B size at least δ, expand with pairing function, constant 0, successor S to create distinct definable elements $S0, SS0, \ldots$. We can use these as indices with pairing function to pick out slices of a universal relation. This interprets the countable signature in a finite language. □

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Proof.

We can simply Skolemize the language, so that submodels in the expanded language are elementary in the original language. □

Bounded size

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Covering reflection for δ is equivalently formulated only for structures B of size at most $2^{<\delta}$.

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If covering reflection fails for B, each $A \in S$ fails to cover some $x_A \in B$. Find $\bar{B} \prec B$ containing every x_A, size at most $2^{<\delta}$. So $\bar{B}$ also fails covering reflection. □

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If covering reflection fails for B, each $A \in S$ fails to cover some $x_A \in B$. Find $\bar{B} \prec B$ containing every x_A, size at most $2^{<\delta}$. So $\bar{B}$ also fails covering reflection. □

Note that $2^{<\delta} = \delta$ is quite common, including every infinite cardinal under GCH.

Covering reflection is Π^1_1

Corollary

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One can refer to all structures B of size at most $2^{<\delta}$ with a second-order quantifier over V_δ, since ${}^{<\delta}2 \subseteq V_\delta$.

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So the covering reflection principle has complexity Π^1_1 over V_δ. □

A hint: not very large?

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The least δ for which covering reflection holds is not weakly compact.

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Proof.

Weakly compact cardinals are Π^1_1-indescribable, and so if they exhibit covering reflection, then there must be a smaller cardinal also exhibiting covering reflection. So the least cardinal δ with covering reflection cannot be weakly compact. $\square$

Another upper bound on size

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The first δ with covering reflection is less than the first Σ_2-correct cardinal. In particular, it is less than the first strong cardinal.

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Proof.

Since Π^1_1 assertions over V_δ are Π_1 in the language of set theory, the existence of a cardinal δ with the covering reflection principle is a Σ_2 assertion. So if there is one, there will be one below the first Σ_2-correct cardinal. In particular, since every strong cardinal is Σ_2-correct, the first cardinal δ with covering reflection will be less than the first strong cardinal. □

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But. . . this is also true of rank-to-rank cardinals, huge cardinals, and more.

A natural weakening

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Definition

The *covering subreflection principle* (CSRP_δ) holds for δ if for every structure *B* in a countable language there is a structure *A* of size less than δ, such that *B* is covered by the elementary images of the elementary submodels of *A*.

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Definition

The *covering subreflection principle* (CSRP_δ) holds for δ if for every structure B in a countable language there is a structure A of size less than δ, such that B is covered by the elementary images of the elementary submodels of A.

That is, for every $b \in B$ there is $\bar{A} \prec A$ and elementary embedding $j : \bar{A} \to B$ with $b \in \mathrm{ran}(j)$.

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For each $b \in B$, pick countable $B_b \prec B$ with $b \in B_b$.

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($\leftarrow$) Consider $\delta > 2^{\aleph_0}$ and B in countable language $\mathcal{L}$.

For each $b \in B$, pick countable $B_b \prec B$ with $b \in B_b$.

Choose family $\{ B_b \mid b \in I \}$ realizing every isomorphism type arising, with I size at most continuum.

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Let $A \prec B$ have $B_b \subseteq A$ for all $b \in I$, size at most continuum.

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Let $A \prec B$ have $B_b \subseteq A$ for all $b \in I$, size at most continuum.

The elementary substructures $B_b \prec A$ for $b \in I$ cover B, as desired. □

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We shall gradually reveal increasingly strong large cardinal lower bounds to the strength of covering reflection.

Extracting strength

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It follows that j must have a critical point, $\mathrm{cp}(j) = \kappa < j(\kappa)$.

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So $P(\kappa) \subseteq A$.

Extracting strength—one measurable.

So we have $j : A \to B = V_{\delta+1}$ with $\kappa = \mathrm{cp}(j)$ and $P(\kappa) \subseteq A$.

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This implies that κ is a measurable cardinal!

We can define the induced normal measure $X \in \mu \leftrightarrow \kappa \in j(X)$ for a fixed $j : A \to B$ with critical point κ.

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If covering reflection holds for δ, then there is a measurable cardinal $\kappa < \delta$.

Perhaps the earlier result that δ itself is not weakly compact was a distraction.

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Furthermore, we get $P(\kappa_1) \subseteq A$ just as we did with κ_0.

Namely, if $X \subseteq \kappa_1$, there is $j : A \to B$ with $\{ \kappa_0, X \} \in \mathrm{ran}(j)$. So both κ_0 and X are in the range of j. So the critical point of j is at least κ_1, and if $X = j(x)$, then x and $j(x) = X$ agree up to κ_1, which implies $X \in A$.

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Conclusion

If covering reflection holds for δ, then there are two measurable cardinals below δ.

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Conclusion

If covering reflection holds for δ, there are infinitely many measurable cardinals below δ.

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Since $P(\kappa_0) \subseteq A$, it follows that $V_{\kappa_0+1} \subseteq A$.

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Further, $j(V_{\kappa_0+1}) = V_{j(\kappa_0)+1}$, since $V_{\delta+1}$ is correct about this.

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So $j \restriction V_{\kappa_0+1} : V_{\kappa_0+1} \to V_{j(\kappa_0)+1}$ is elementary, with crit point κ_0.

Pushing even harder—1-extendible

For even more strength, let us show κ_0 is far more than merely measurable.

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Conclusion

If covering reflection holds for δ, then there is a 1-extendible cardinal below δ.

Extendible cardinals

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Stronger than supercompact, in the upper realms of the large cardinal hierarchy.

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Conclusion

The consistency strength of covering reflection exceeds a supercompact cardinal.

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If $\beta \leq \kappa_\beta$, which is true already for a long way, then all initial segments of $\langle \kappa_\alpha \mid \alpha < \beta \rangle$ are also in the range of the $j : A \to B$ witnessing κ_β. So this embedding is also relevant when defining previous κ_α, and consequently $\kappa_\alpha \leq \kappa_\beta$ for all $\alpha < \beta$.

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But since those κ_α are in $\mathrm{ran}(j)$, but κ_β is not, it follows that $\kappa_\alpha < \kappa_\beta$ for all $\alpha < \beta$.

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Let $\lambda = \kappa_\gamma$ when this occurs. So λ is critical point of some $j : A \to B$ with $\langle \kappa_\alpha \mid \alpha < \gamma \rangle$ in $\mathrm{ran}(j)$.

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This *j* is relevant for $\alpha < \lambda$, so $\kappa_\alpha < \lambda$. Thus, $\lambda = \sup_{\alpha<\lambda} \kappa_\alpha$.

Each κ_α is λ-extendible by the reasoning we gave earlier.

Conclusion

If covering reflection holds at δ, then there is measurable $\lambda < \delta$ that is a limit of λ-extendible cardinals.

A little more

In fact, in the paper we prove that unboundedly many of the κ_α for $\alpha < \lambda$ are extendible inside V_λ.

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Conclusion

If covering reflection holds at δ, then there is measurable cardinal λ below δ such that V_λ has a proper class of extendible cardinals.

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$V_{\bar\lambda}$ will have a proper class of extendible cardinals, with λ as an extendible limit of extendible cardinals inside it.

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$V_{\bar{\lambda}}$ will have a proper class of extendible cardinals, with λ as an extendible limit of extendible cardinals inside it.

Conclusion

If covering reflection holds for δ, then it is consistent to have a proper class of extendible cardinals and extendible limits of extendible cardinals, and limits of limits, and so forth.

Upper bound?

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Let me do so now.

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This is a large cardinal in the upper realm of large cardinals, above supercompact, extendible, and so forth, but below rank-to-rank.

Theorem

If κ is huge, then the covering reflection principle holds of κ. The least cardinal δ exhibiting covering reflection is therefore strictly less than κ.

Proof of hugeness upper bound

Assume that κ is huge, witnessed by $j : V \to M$.

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By hugeness, M and V have same substructures of $j(B)$ of size $j(\kappa)$, and same embeddings into $j(B)$.

So $j(B)$ is also a counterexample to covering reflection for $j(\kappa)$ in V.

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And it hits $j(x)$. Contradiction.

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A cardinal κ is an *anchor* cardinal if for every $X \subseteq V_\kappa$ there is $\kappa_0 < \kappa_1 < \kappa$ and elementary embedding $j : \langle V_{\kappa_1}, \in, X \cap V_{\kappa_1} \rangle \to \langle V_\kappa, \in, X \rangle$ with $\kappa_0 = \mathrm{cp}(j)$ and $j(\kappa_0) = \kappa_1$.

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Related to *links* and *chains* in [SRK78].

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Related to *links* and *chains* in [SRK78].

Every huge cardinal has a normal measure concentrating on anchor cardinals.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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References I

[Ham+] Joel David Hamkins, Nai-Chung Hou, Andreas Lietz, and Farmer Schlutzenberg. "The covering reflection principle". (). in preparation.

[Ham23] Joel David Hamkins. *A Löwenheim–Skolem–Tarski-like property*. MathOverflow answer. 2023. https://mathoverflow.net/q/458891 (version 22 November 2023).

[Hou23] Nai-Chung Hou. *A Löwenheim–Skolem–Tarski-like property*. MathOverflow question. 2023. https://mathoverflow.net/q/458852 (version 21 November 2023).

[Lie23] Andreas Lietz. *A Löwenheim–Skolem–Tarski-like property*. MathOverflow answer. 2023. https://mathoverflow.net/q/458929 (version 23 November 2023).

[Sch23] Farmer Schlutzenberg. *A Löwenheim–Skolem–Tarski-like property*. MathOverflow answer. 2023. https://mathoverflow.net/q/458904 (version 22 November 2023).

References II

[SRK78] Robert M. Solovay, William N. Reinhardt, and Akihiro Kanamori. "Strong axioms of infinity and elementary embeddings". *Ann. Math. Logic* 13.1 (1978), pp. 73–116. ISSN: 0003-4843. DOI: 10.1016/0003-4843(78)90031-1.