The ontology of mathematics

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We should distinguish between the number and the numeral, between the mathematical object and the description of that object.

The largest number

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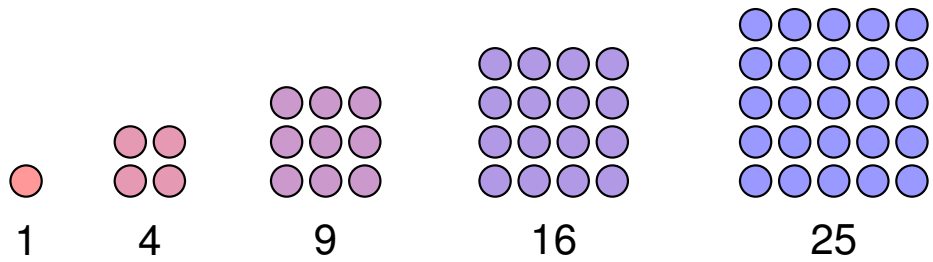
Broken numbers from the mine were used for fractions.

Kinds of numbers

Square numbers can be arranged in a square.



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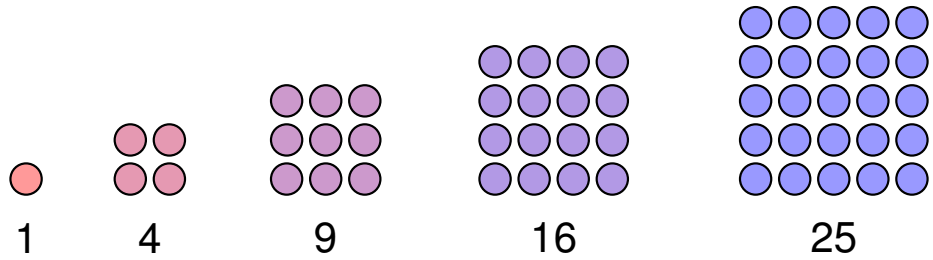
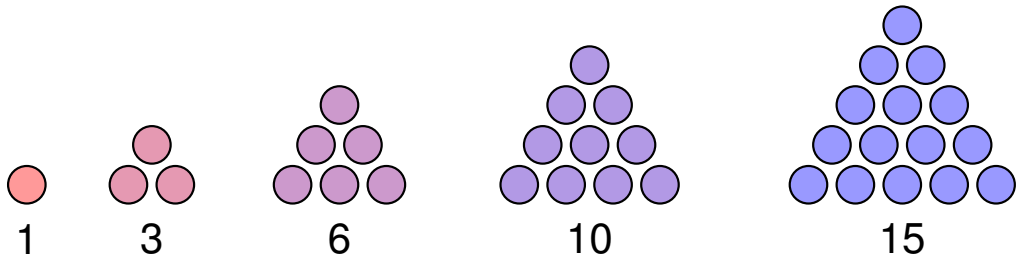


Triangular numbers can be arranged in a triangle.



One proceeds to the *hexagonal* numbers and so forth.

Palindromic numbers

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121 523323325

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Every number is a palindrome in any sufficiently large base. It becomes a single digit.

A confounding case

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The first is in lowest terms; the second is not.

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Being in lowest terms is not a property of numbers, but of representations of numbers, of numerals.

What is a number?

Philosophers of mathematics describe a variety of philosophical attitudes to take towards the nature of mathematical existence.

- platonism
- formalism
- intuitionism
- constructivism
- logicism
- structuralism
- nominalism
- fictionalism

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In the platonic realm, there are ideal forms, perfect lines and circles.

And numbers, functions and sets.

Structuralism

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The integer number zero, for example, is the additive identity in the ring of integers.

It is also the only additive integer idempotent, the only number z for which $z + z = z$.

Dedekind arithmetic

For a structuralist account of the natural numbers

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- Every number is generated from 0 by successor. That is, if A is a set of numbers, with $0 \in A$ and $x \in A \implies S(x) \in A$, then A has all numbers.

All the familiar arithmetic truths follow as consequences.

Categoricity

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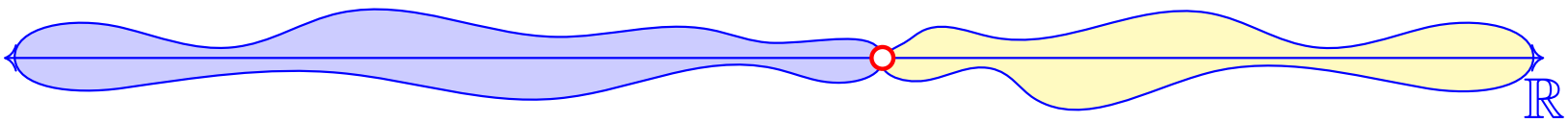
Categoricity entitles us to refer to $\mathbb{N}$, as a unique structure, by saying merely that they fulfill Dedekind axioms.

Important origin for structuralism.

What is a real number?

Dedekind observed how every real number is determined by the cut it makes in the rational numbers.



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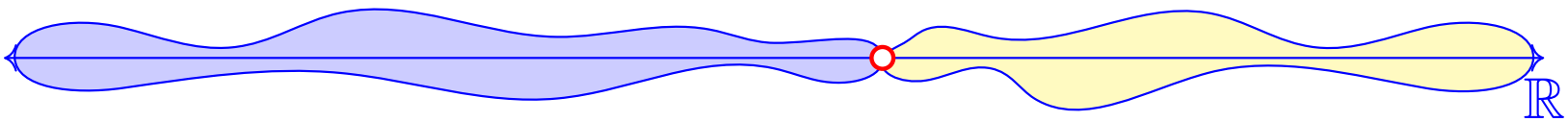

The real numbers are *Dedekind complete*: every cut is filled.

> *And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous;*
>
> *[Ded01, §3, Continuity and Irrational numbers, 1872]*

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> *The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil. [Rus19, p. 71]*

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He undertakes the 'honest toil' of placing an ordered field structure on the set of Dedekind cuts, and proving that they thereby become Dedekind complete.

On this account, a real number is a Dedekind cut.

Cauchy completion

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Categoricity for the real numbers

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This is a categorical characterization of the real numbers.

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For the structuralist, it doesn't matter.

The reals are any complete ordered field, and it doesn't matter which one, since they are all isomorphic.

Every real number plays a distinct structural role in the real field.

Structuralism in the reals

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In any complete ordered field, every rational number is definable, and every real number is characterized by how it cuts the rational numbers.

Complex numbers

Given the real numbers, one proceeds to the complex numbers $\mathbb{C}$, motivated by the enticing, yet perhaps terrifying, possibility of the imaginary unit

$$i = \sqrt{-1}.$$

We want to consider complex numbers of the form $a + bi$, where a and b are real.

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Perhaps my i is your $-i$. How would we know?

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This extra coordinate structure, beyond the field structure, distinguishes i from $-i$.

Complex numbers in Heaven

Imagine that at your death you are astonished to meet God in Heaven, who informs you, *"Yes, you were completely right about platonism for the real numbers—there they are!"*

He points across the way, and indeed there you see them, the real numbers, each of them a perfect platonic ideal of its kind. You find the numbers π, e, $\sqrt{2}$, each where you expect them.

"But," God continues, *"you were wrong about platonism for the complex numbers; you have to construct them from the reals as pairs* (a, b)*, with the parentheses and comma and everything."*

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Something absurd about this. Opens the door to *plenitudinous platonism*.

Categoricity: based upon set theory

Curious observation: categoricity arguments for $\mathbb{N}$ and $\mathbb{R}$ take place in set theory.

- Natural numbers $\mathbb{N}$ characterized by induction principle, concerned with arbitrary $A \subseteq \mathbb{N}$.
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Is this a problem for structuralism?

How can we secure a definite number concept upon the comparatively murky concept of ‘arbitrary’ set?

Set theory as a foundation

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Gradually, the capacity of set theory to express essentially any abstract mathematical structure emerged.

Set theory began to serve as an ontological foundation for mathematics.

To be precise in twentieth-century mathematics often came to mean to specify one's mathematical structure set-theoretically.

Paradise

In light of the power and utility of set theory, Hilbert had said,

> *No-one shall cast us from the paradise that Cantor has created for us.*

Set theory as uniform framework

Rise of set theory as a unifying foundation was historically important

Mathematics diversified, applying results from one area to another.

Notions from topology and analysis used in algebra or conversely.

This is logically incoherent without a common framework.

Set theory provided a unified context, facilitating the practice.

Focal question

Let me focus on the question:

> *Does every mathematical problem have a definite answer?*

In particular,

> *Does every set-theoretic assertion have a definite truth value?*

The iterative conception

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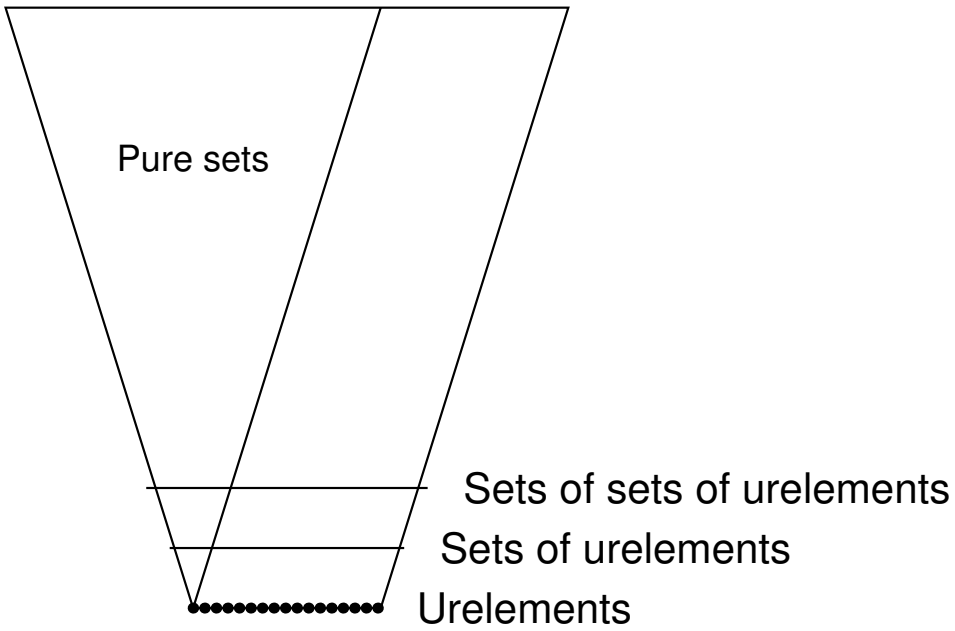
A vast hierarchy emerges...



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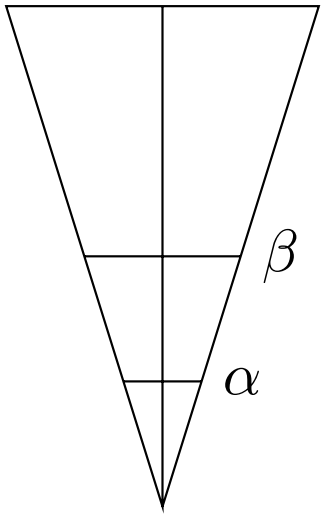
Every mathematical structure that arises in set theory can be found isomorphically within the pure sets.

According to structuralism, therefore, the pure sets work just as well as a foundation of mathematics.

The cumulative universe

Cumulative universe of pure sets

Contains copies of every conceivable mathematical structure, the entirety of mathematics.



The Universe View

The *universe view* asserts:

There is a unique absolute concept of set, giving rise to the cumulative universe of all sets.

The universe view has determinate set-theoretic truth, and hence also determinate mathematical truth.

Every mathematical question has a determinate answer.

Universe view via consequentialism

Support for the universe view based on consequences.

Attractive and explanatory regularity features follow from strong set-theoretic axioms, particular from the large cardinal axioms, strong axioms of infinity.

This is offered as evidence for the truth of those axioms.

Main challenge for the universe view

The central discovery in set theory

Enormous range of set-theoretic possibility. Powerful tools for constructing alternative set-theoretical universes.

forcing, ultrapowers, canonical inner models, etc.

Much of set-theory is about constructing diverse models of set theory, with diverse truths.

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A major open question from the time of Cantor.

Number 1 problem on Hilbert's list.

Independence of the continuum hypothesis CH

Decades after Cantor struggled with CH...

- (Gödel 1938) CH holds in the constructible universe L.
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Perhaps this is simply the weakness of our axioms?

We seek philosophical grounds to settle the CH question.

Imaginary alternative history

Imagine an alternative history for set theory:

- Imagine that theorems were increasingly settled in the base theory.
- ...that independence phenomenon was limited to paradoxical meta-logic statements.
- ...that independence results were usually settled by missing natural axioms.
- ...that the basic structure of set theory became increasingly stable and agreed-upon.

That would have been evidence for the universe view.

But the actual history is totally different...

Actual history: an abundance of universes

Set theorists have discovered a vast diversity—a chaotic jumble of set-theoretic possibility.

Would you like CH or $\neg\mathrm{CH}$? How about $\mathrm{CH} + \neg\diamondsuit$? Do you want $2^{\aleph_n} = \aleph_{n+2}$ for all n? Suslin trees? Kurepa trees? Martin's Axiom?

Set theorists build models to order.

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Amazing fact

Continuum hypothesis comes out as true, or false, depending on choice of $\mathbb{B}$.

Many statements in set theory can be changed by working with a different $\mathbb{B}$.

Pluralism in the set-theoretic foundations

As a result, the fundamental object of study in set theory has become: the model of set theory.

We have diverse set-theoretic universes.

Set theory has discovered an entire cosmos of set-theoretical universes, connected by forcing or large cardinal embeddings, like lines in a constellation filling a dark night sky.

A challenge to the universe view

Challenge for the universe view

How to explain the enormous diversity of set-theoretic possibility?

Will the alternative universes be explained away as imaginary?

Not sufficient to argue that one has a robust theory, if competing alternatives appear set-theoretic.

Universists must explain (or explain away) the pluralistic illusion.

The Multiverse View

Competing view accepts alternative set concepts as fully real.

The Multiverse view. The philosophical position holding that there are many set-theoretic universes.

Numerous distinct concepts of set, each giving rise to a set-theoretic universe.

A philosophical enterprise becomes mathematical

Many of these concepts of set are closely enough related to be analyzed together from the perspective of a single set concept.

So what might have been a purely philosophical enterprise—comparing different concepts of set—becomes in part a mathematical one.

And the subject known as the philosophy of set theory thus requires a pleasing mix of (sometimes quite advanced) mathematical ideas with philosophical matters.

Multiverse view is realism

The multiverse view is a brand of *realism*. Alternative universes each fully as real as the Universe of sets on the Universe view.

Our mathematical tools—forcing, etc.—allow us to glimpse into other mathematical worlds, providing evidence they exist.

Multiverse view is platonist, namely, platonism about universes.

Plenitudinous platonism

The multiverse view has strong affinities with Mark Balaguer's view:

> "The version of platonism that I am going to develop in this book—I will call it *plenitudinous platonism*, or alternatively, *full-blooded platonism* (FBP for short)—differs from traditional versions of platonism in several ways, but all of the differences arise out of one bottom-level difference concerning the question of *how many* mathematical objects there are. FBP can be expressed very intuitively, but also rather sloppily, as the view that *all possible mathematical objects exist*." (p. 5)

Strong analogy with Geometry

Geometrical concepts—points, lines, planes—clear, absolute.

Shattered with non-Euclidean geometry to distinct kinds of geometrical universes.

At first, simulate non-Euclidean geometry inside Euclidean geometry (for example, ‘line’ = great circle on sphere).

In time, non-Euclidean geometry accepted as fully real.

Why not also for set theory?

The dream solution template for CH

Set theorists yearn for a definitive solution to CH.

Step 1. Produce a new, natural set-theoretic axiom Φ.

Step 2. Prove that Φ settles CH.

Thus, CH would be settled. Everyone would accept Φ and its consequences.

Dream solution will never be realized

I argue the dream solution is unworkable.

Rich experience in worlds with CH and others with $\neg\mathrm{CH}$.

Seems fully set-theoretic—prevents us from accepting a new axiom that settles CH.

My prediction

Any specific dream solution proposal will be rejected from a position of deep mathematical experience with the contrary.

Dream solution will never be realized

I argue the dream solution is unworkable.

Rich experience in worlds with CH and others with $\neg\mathrm{CH}$.

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My prediction

Any specific dream solution proposal will be rejected from a position of deep mathematical experience with the contrary.

This was exactly the reaction to Freiling's [Fre86] axiom of symmetry.

CH in the Multiverse

CH is independent, yes, but more than this...

Every set-theoretic universe has a forcing extension in which CH holds, and another in which it fails.

We can turn CH on and off like a lightswitch.

We have a rich experience in the resulting models, which seem fully set-theoretic to us.

Proper attitude to CH

So it is not merely that CH is formally independent and we have no additional knowledge.

Rather, we have an informed, deep understanding of *how* CH could hold or fail, of how to build such worlds from one another.

If a proposed axiom Φ settles CH, then we will not look upon it as intrinsic to the concept of set, since we already know how it can fail.

It would be like someone having an axiom implying that only Brooklyn existed, while we already know about Manhattan and the other boroughs of New York.

Thus, the dream solution will not succeed.

The CH is settled

On the multiverse perspective, the CH question is settled.

It is incorrect to describe it as an open question.

The answer consists of our detailed understanding of how the CH both holds and fails throughout the multiverse, of how these models are connected and how one may reach them from each other while preserving or omitting certain features.

Fascinating open questions about CH remain, of course, but the most important essential facts are known.

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Fine, we can neither prove nor refute the continuum hypothesis; it is independent of the ZFC axioms.

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But still, mathematicians want to know: is CH true?

The question may seem innocent, but interpreting it sensibly leads to the extremely difficult philosophical issues I have been discussing in this talk.

Ultimately, the question becomes: do we have just one mathematical world or many?

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Thank you.

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