Truth and Paradox in the Theory of Finite and Infinite Games

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

33rd annual Owens Memorial Lecture
Wayne State University
April 16, 2026

WAYNE STATE UNIVERSITY
Department of Mathematics

The 33rd Annual
Owens Lecture

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Abstract

Let us explore the theory of finite and infinite games, from the hypergame paradox to the fundamental theorem of finite games, which generalizes to vast classes of infinite games. We shall see how the ideas play out in infinite chess, infinite draughts (checkers), infinite Hex, infinite Wordle, and many other games.

April 16, 2026, 2:30pm, 1507 Engineering Building

Coffee and snacks afterward in Room 1146, Faculty/Administration Building

Joel David Hamkins

Joel David Hamkins is a mathematician and philosopher specializing in the mathematics and philosophy of the infinite. His research spans topics in mathematical and philosophical logic and the philosophy of mathematics, from the strong axioms of infinity to infinitary computability and infinite game theory. He is the O'Hara Professor of Logic at the University of Notre Dame and formerly Professor of Logic at Oxford University.

Owens Lecture

The Owens Lecture is named for the late Owen G. Owens, a former Professor of Mathematics at Wayne State University. The Lecture is supported by the Owens Fund, which was established by the family of Professor Owens.

The Logic Bar

Three logicians walk into the Logic Bar.

Waiter: *Do you all want beer?*

First logician: *I don't know.*

Second logician: *I don't know.*

Third logician: *Yes.*

Question

How many beers does the waiter bring?

The Blue-Eyed Islanders

A remote island has 100 perfectly logical inhabitants, each with vivid deep blue eyes.

They never discuss eye color. Further, if any of them should learn their own eye color, they leave the island next dawn in a flashy display.

A trusted friend visits the island. At her departure, in front of the entire community, she remarks:

At least one of you has blue eyes.

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Exactly 100 days later, all the islanders depart at once in a flashy display.

Why?

Blue-eyed islanders solution

Work up from easier instances.

One islander. If there is only one blue-eyed islander, he learns his eye color immediately and departs next dawn.

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By induction, n blue-eyed islanders depart at dawn of day n.

All 100 islanders depart at dawn 100th day in a flashy display.

A subtle philosophical conundrum

Observation

When the visitor says

"At least one of you has blue eyes,"

this is something that each islander knows already.

They didn't seem to learn any new information.

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Question

How is it possible that being told something they already knew enabled them to deduce their own eye color?

The Pirate-Treasure Division Problem

Pirate ship with a fearsome, perfectly logical crew. The pirate treasure of 100 gold coins is to be divided.

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Pirate-treasure division protocol. Pirates are ranked—captain, first lieutenant, second lieutenant. Let's call them pirate 1, pirate 2, etc.

- Lowliest pirate mounts plank, makes proposal.
- All pirates vote.
- If majority approve, that's the plan.
- If not, the pirate walks the plank. Next lowliest pirate mounts plank.

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Question

You are pirate 10. What is your proposal?

Pirate value system

Strictly ordered with priority.

- A Most important, stay alive.
- B Get gold—the more the better.
- C Cause the death of other pirates, if possible.
- D All other things being equal, arrange when possible that gold goes to senior pirates.

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Three pirates. Pirate 3 needs 2 votes. Pirate 2 will vote in favor, since otherwise dead. So pirate 3 proposes, “I get all the gold.” Approved by 2,3.

Four pirates.

Four pirates. Pirate 4 needs three votes.

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Five pirates. Needs three votes. Can buy pirate 3’s vote for 1 gold.

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Five pirates. Needs three votes. Can buy pirate 3’s vote for 1 gold. Can buy pirate 1’s or 2’s vote with 2 gold.

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Five pirates. Needs three votes. Can buy pirate 3’s vote for 1 gold. Can buy pirate 1’s or 2’s vote with 2 gold. In light of pirate value D, prefer pirate 1. So: 2 gold to 1; 1 gold to 3; and 97 gold to me. Passes.

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	1	2	3	4	5	6	7	8	9	10
One pirate	100									
Two pirates	$*$	X								
Three pirates	0	0	100							
Four pirates	1	1	0	98						
Five pirates	2	0	1	0	97					
Six pirates	0	1	2	1	0	96				
Seven pirates	1	2	0	0	1	0	96			
Eight pirates	2	0	1	1	0	1	0	95		
Nine pirates	0	1	2	0	1	0	1	0	95	
Ten pirates	1	2	0	1	0	1	0	1	0	94

The Philosophical society

Annual entrance examination for 100 ambitious philosopher candidates to the Philosophical Society.

We all stand in a line facing the society hall.

A hat, either black or white, is placed on each head. We can see only those ahead, but not those behind or our own.

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If we are all correct, then we are admitted as a group, but otherwise all are sent down.

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A consession: before it starts, the candidate at very back may inspect her hat and even change the color if desired.

How well can we do?

First attempts

Go with the majority

Everyone guesses the majority color they see, otherwise white.

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Everyone flip a coin for the guess. This will get half right on average, but likelihood of everyone right is vanishingly small.

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Everyone flip a coin for the guess. This will get half right on average, but likelihood of everyone right is vanishingly small.

Guaranteed 51 or more

Odd-position people say the color of the person just in front of them; that person then says same color. This is at least half right. Plus very first person. So at least 51 correct.

Can we do better?

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If this is even, then adopt a white hat, otherwise black. Announce accordingly.

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Each person pays attention to the prior announcements, and looks ahead. Knowing the parity, they can deduce their own hat color.

Everyone will be right!

The Physics Society

The physicists use 10 hat colors, instead of just two.

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Simply use base 10 instead of base 2. Each hat is a number 0 to 9, and first candidate picks the total sum mod 10.

Everyone can determine their own hat color based on what came before and the total residue.

So everyone will be correct!

The mathematical society

The mathematicians use natural numbers, written on slates.

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Easy solution: first person simply announces the total sum.

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What about real numbers?

The mathematical society

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Everyone can calculate their number based on the prior numbers, the observed numbers, and the total sum.

What about real numbers?

Same method works in the real numbers. Indeed, it works in any group.

Since any nonempty set can carry a group structure, the method works with any set of labels!

The Sudoku game

5			4			1		
3					5			7
	9				3	5		
2			7					
		4			8			
6								9
		6				4		
		1			9	2		
4				5			8	

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- Consider Sudoku as a two-player game
- Start from empty board, perhaps very large
- Players take turns placing numbers
- Plays must obey the Sudoku condition—no repeats on any row, column, or subboard

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- Consider Sudoku as a two-player game
- Start from empty board, perhaps very large
- Players take turns placing numbers
- Plays must obey the Sudoku condition—no repeats on any row, column, or subboard
- A player loses when there is no legal move
- So this version of the game is not about a global solution

Sudoku game, even size board

Second player has a winning strategy.

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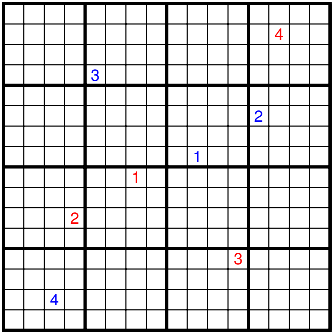

Strategy: Second player copies, reflected through the center.

Key observation: if previous move was legal, so is reflection.

So the second player wins.

Sudoku game, odd boards

What about odd-sized boards?

	9				5			
							2	
			4					
		3		5		7		
					6			
	8							
			5				1	

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	9				5			
							2	
			4					
		3		5		7		
					6			
	8							
			5				1	

Now, it is a first-player win.

Start with 5 at center, then reflect with dual number.

Key observation: if previous move was legal, so is reflection.

So first player will win.

Infinite Sudoku

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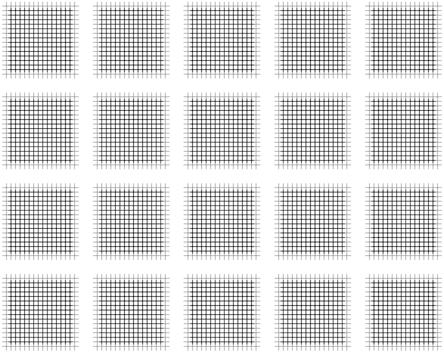

The board is an infinite array of infinite Sudoku subboards.

Each subboard is like the integer grid.

Infinite Sudoku

With Infinite Sudoku, natural to use Solver/Spoiler rules of play.

- The Solver aims to create a full, global solution.
- The Spoiler aims to prevent this.

Can the Spoiler set traps and create obstacles so as to prevent the Solver from succeeding in the limit?

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The Solver, I claim, can win infinite Sudoku.

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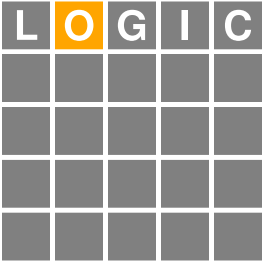
- Solver can ensure that every cell is filled.
- Solver can ensure that every integer appears once on every row.
- Solver can ensure that every integer appears once on every column.
- Solver can ensure that every integer appears once on every subboard.

Solver can organize the requirements so as to fulfill them all.

Thus, the Solver has a winning strategy.

Wordle



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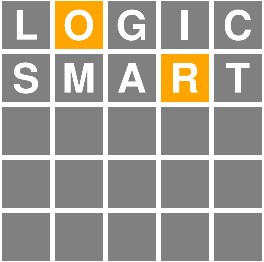

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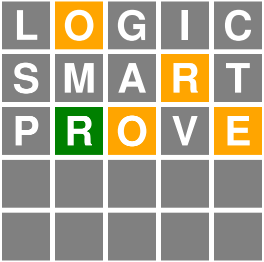

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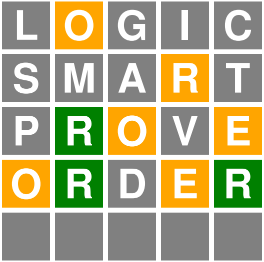
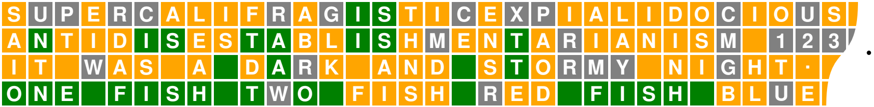


Wordle

Long Wordle

I should like to greatly enlarge the scale of the game.



Consider a variation of Wordle using very long words or whole phrases, perhaps book-length passages—perhaps millions of letters!

Furthermore, let us consider any crazy fixed dictionary of allowed phrases, perhaps including many nonsense sequences of letters.

How long to win Long Wordle?

Suppose we are playing Long Wordle, with phrases having millions of letters each.

Question

How many steps will it take to win?

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With 26 letters, there are $26^{1000000}$ many distinct “words” of length one million.

If we wrote down one million of them every second since the beginning of time at the Big Bang, we will have hardly started!

Long Wordle theorem

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Iterate. Since 26 letters, this can go on at most 26 steps. □

The infinite Wordle theorem

Same idea applies to *infinite* Wordle, allowing infinitely long words.

Infinite Wordle theorem

In the game of infinite Wordle with any dictionary and a finite alphabet of size n, we can win in at most n steps.

The analysis leads eventually to deeper issues with infinite alphabets and transfinite play.

Related to Infinite Mastermind. Things get very interesting and complicated.

Hypergame and the finite-play games

A game is a *finite-play game* when it has no infinitely long play—all plays of the game come to a conclusion in finitely many moves.

Nim, Hex, Connect Four, Tic Tac Toe, and so forth.

What about Chess?

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What about Chess? Three-fold repetition rule, 50-move rule make chess a finite-play game.

Let's play Hypergame

First player chooses a finite-play game.

Then, we play that game. The outcome of the game-in-a-game determines the hypergame outcome.

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Then, we play that game. The outcome of the game-in-a-game determines the hypergame outcome.

Observation

All plays of hypergame will come to a conclusion in finitely many moves.

Thus, hypergame itself seems to be a finite-play game.

Playing Hypergame

A mischievous First player opts for hypergame itself.

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Flirtatious second player opts for... hypergame.

Hypergame, hypergame, hypergame, hypergame, ...

But that is an infinite play!

The Hypergame paradox

On the one hand

Since the first player must choose a finite-play game, every play of hypergame will end in finitely many moves. So Hypergame is a finite-play game.

The Hypergame paradox

On the one hand

Since the first player must choose a finite-play game, every play of hypergame will end in finitely many moves. So Hypergame is a finite-play game.

On the other hand

Since Hypergame is a finite-play game, the first player can opt for it, and so forth:

hypergame, hypergame, hypergame, . . .

But this is an infinite play!

How do we resolve the contradiction?

Abstract theory of strategies

Let's develop a little of the abstract theory of games and strategies.

- What is a game?
- What does it mean to have a strategy in a game?
- What is a winning strategy?

Abstract theory of strategies

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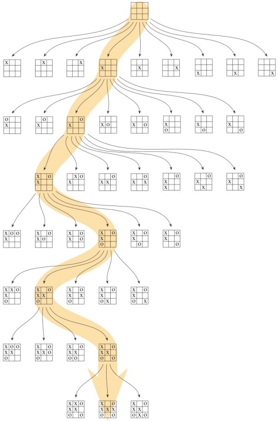
- What is a game?
- What does it mean to have a strategy in a game?
- What is a winning strategy?
- Does every game have a winning strategy?
- Does every infinite game have a winning strategy?

Some of these questions begin to engage with deep mathematical and sometimes philosophical questions.

Game tree

Every game gives rise to a *game tree*, the tree of all possible positions of the game. Plays of the game are paths through the game tree.



Fundamental theorem of finite games

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In every finite two-player game of perfect information, one of the players has a winning strategy.

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Several proofs

- Game tree back propogation
- Avoid losing—if opponent has no winning strategy, play so as to preserve this.
- Induction on size of game tree
- De Morgan law
- Ordinal game values

First proof—back-propagation

Consider the game tree T of a finite game.

Terminal nodes labeled as a win for one player or the other.

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- A node gets a player's label if they can move to a child node with their label.

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Terminal nodes labeled as a win for one player or the other.

Propagate the winning information up the tree. Working from bottom, label a node if a player can win from that node.

- A node gets a player's label if they can move to a child node with their label.
- A node gets opponent's label if all moves lead to a child node with that label.

The root node will get one label or the other, and whoever it is can win—play to stay on your labels. □

Second proof—avoid losing

Let W be set of nodes from which first player can win.

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If this includes the root node, then first player can win.

Second proof—avoid losing

Let W be set of nodes from which first player can win.

If this includes the root node, then first player can win.

Otherwise, second player can play so as to avoid those nodes.

Key point: if all your moves lead to a node from which first player wins, then first player already could win from your node.

So second player wins by avoiding the losing nodes. □

Third proof—by De Morgan law

Consider any finite game.

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Player 2 has a winning strategy means:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_n \exists y_n \text{ (the play } \vec{x}\vec{y} \text{ is a win for player 2)}$$

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Player 2 does *not* have a winning strategy is

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By De Morgan law, this is same as

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \exists x_n \forall y_n \left(\text{the play } \vec{x}\vec{y} \text{ is a win for player 1}\right)$$

This asserts precisely that player 1 has a winning strategy! □

Generalization to games with draws

Fundamental theorem of finite games (with draws)

In any finite game of perfect information, allowing draws, either one of the players has a winning strategy or both players have draw-or-better strategies.

Follows from the win/loss version.

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Chess

For the game of chess, either one of the players white or black has a winning strategy or both players have draw-or-better strategies.

Open determinacy

Similar ideas work for *open* games, which are games for which all winning plays for one of the players are known as wins at a finite stage of play.

Open determinacy theorem (Gale-Stewart 1953)

In every open game, one of the players has a winning strategy (or both have drawing strategies).

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Open determinacy theorem (Gale-Stewart 1953)

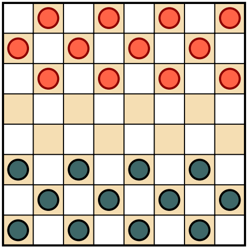
In every open game, one of the players has a winning strategy (or both have drawing strategies).

One way to prove this uses the concept of *transfinite ordinal game values*.

These generalize the chess idea of mate-in-2 or mate-in-3, but work with infinite games.

Finite draughts

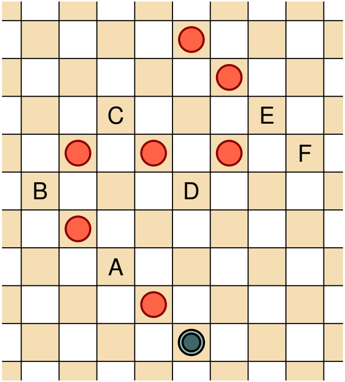
The familiar game of draughts, also known as checkers.



Although often played by children, draughts admits serious advanced play, with draughts grandmasters competing in international tournaments.

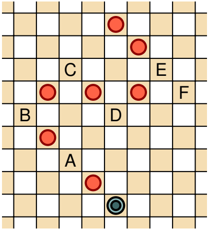
Infinite draughts

Let us endeavor instead to play *infinite* draughts.



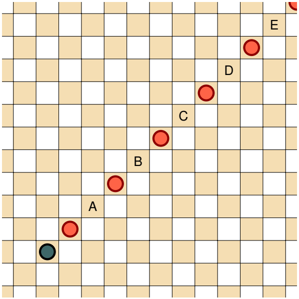
The checkerboard extends without end in every direction.

The rules



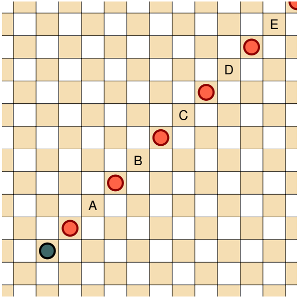
- No standard starting configuration.
- Analyze the game from any given position.
- Pawns and kings.
- Obligatory jumping rule.
- Obligatory iterated jumping.
- Winning condition: you lose when you have no legal move.

Infinite iterations



During an infinite iterated jump, the red pieces are all removed.

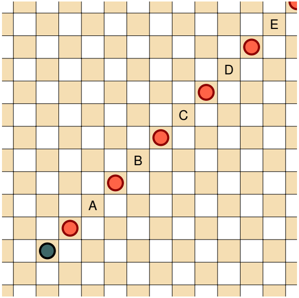
Infinite iterations



During an infinite iterated jump, the red pieces are all removed.

A conundrum: what about the black piece?

Infinite iterations

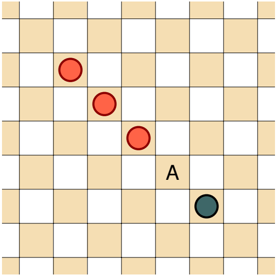


During an infinite iterated jump, the red pieces are all removed.

A conundrum: what about the black piece?

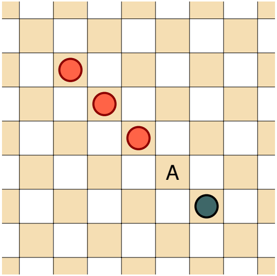
Infinite iterated-jump rule: the jumping piece also is removed.

Game value 2



This position, with Red to play, has game value 2 for Red.

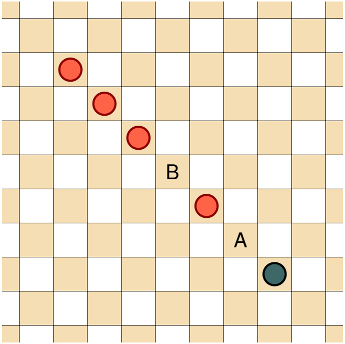
Game value 2



This position, with Red to play, has game value 2 for Red.

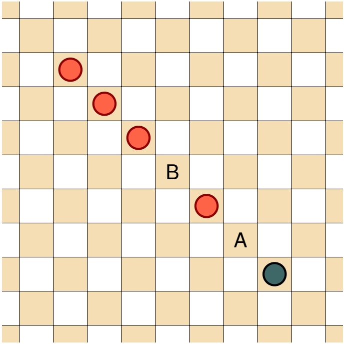
Red can advance leading pawn to A as bait, obligating Black to jump, after which Red recaptures.

Game value 3



Red to play, with game value 3.

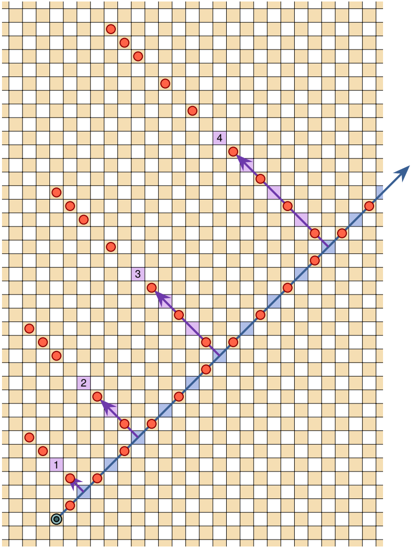
Game value 3



Red to play, with game value 3.

Red advances isolated pawn to A, obligating Black to jump, then advance to B, Black jumps, then Red captures.

Game value ω



Black is obligated to jump.

Can jump infinitely,
but that is an immediate loss.

So will come
to rest at some square *n*.

And then lose in *n* moves.

This is game value ω.

Infinite draughts exhibits high game values

Theorem (Hamkins & Leonessi 2022)

Every countable ordinal arises as the game value of a position in infinite draughts.

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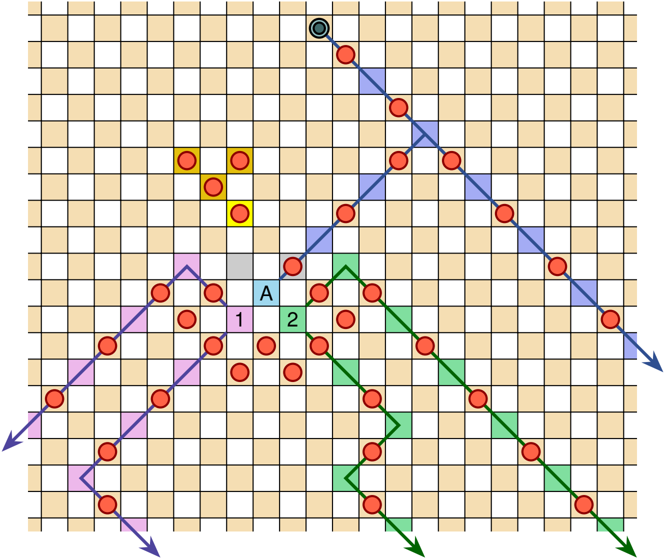
Main proof idea: embed certain well-founded trees into positions of infinite draughts.

The draughts play will unfold in a manner as though Black is climbing the tree, and consequently the game value will track the ordinal rank of the well-founded tree itself.


A
1
2

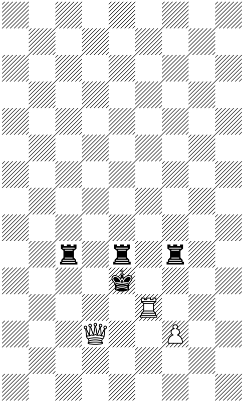
Infinite chess

Let us turn now to infinite chess.

Infinite chess is a game of the mind—we do not sit down in a café and play infinite chess over espresso.

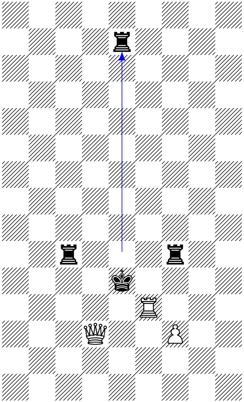
But rather, we sit down in that café and wonder what it would be like to play from this position or that one.

A finite position with value ω



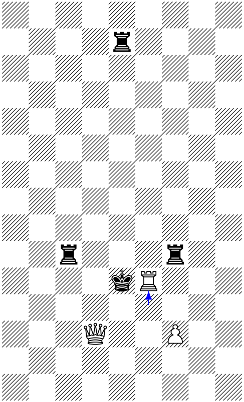
Black to move.

A finite position with value ω



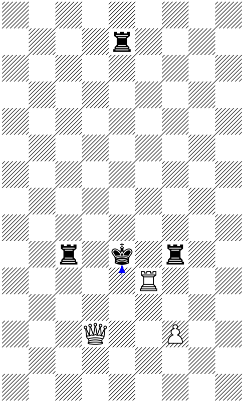
Black moves up arbitrary height

A finite position with value ω

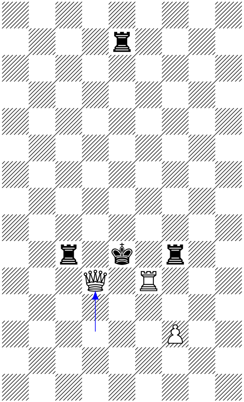


Check

A finite position with value ω



A finite position with value ω



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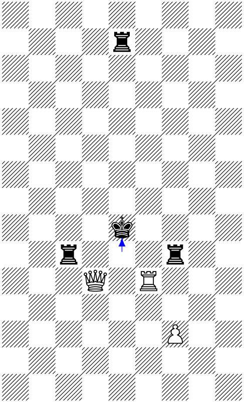

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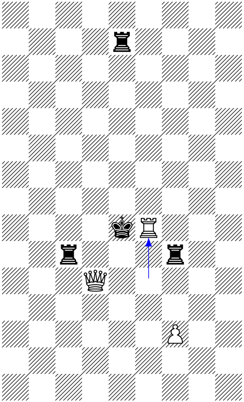

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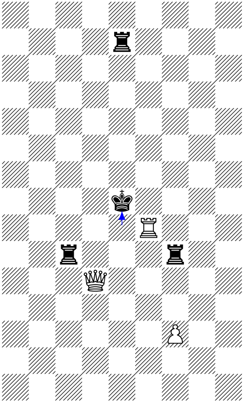

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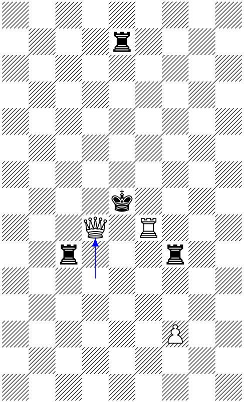

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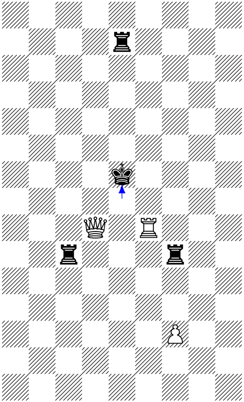

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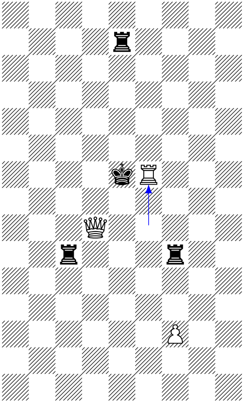

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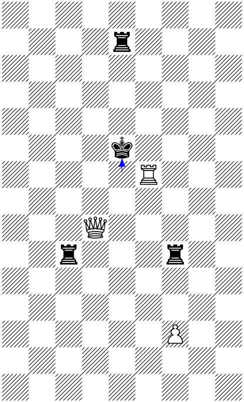

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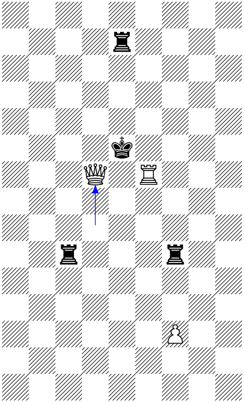

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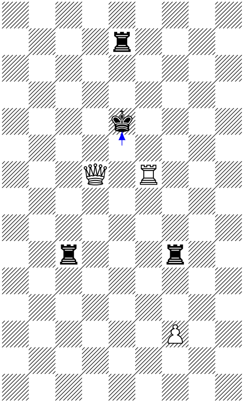

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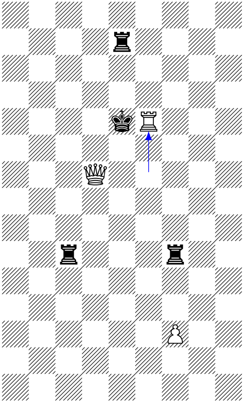

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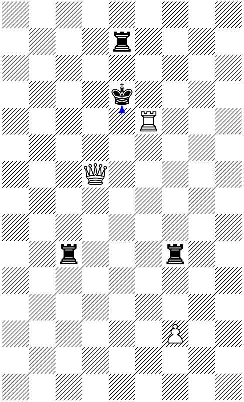

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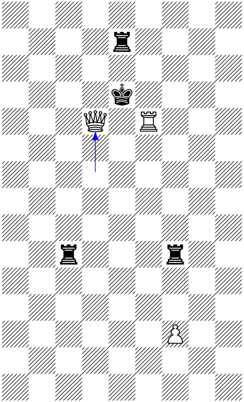

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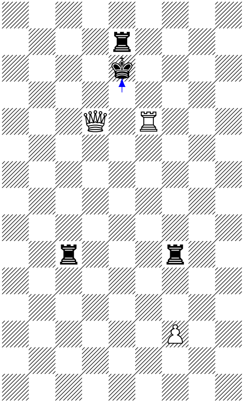
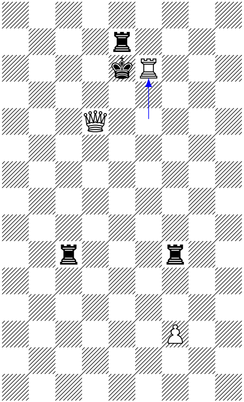
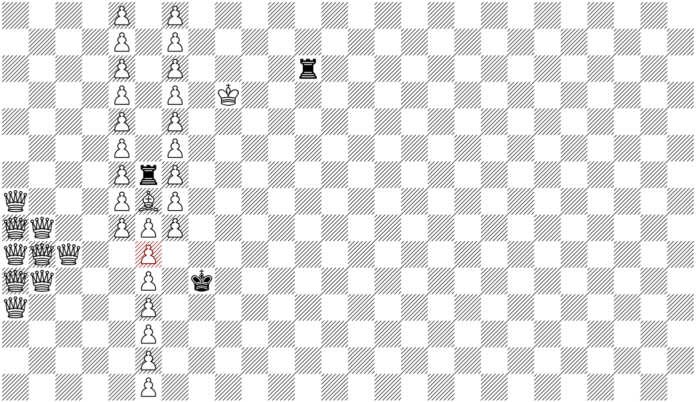


A finite position with value ω



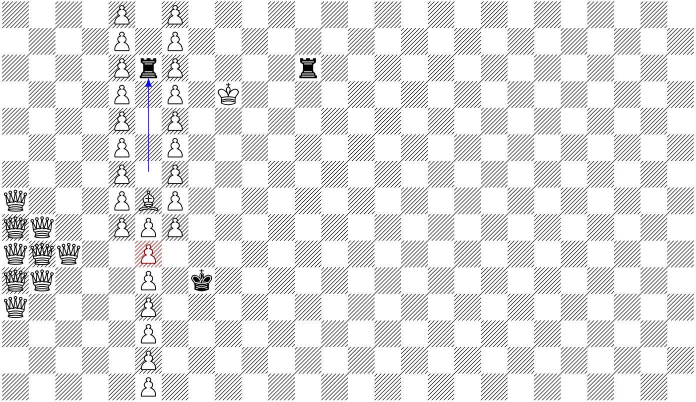
Checkmate. Black can cause arbitrary delay, but the only choice is on first move, so the initial value is ω.

Releasing the Hordes, with value ω^2



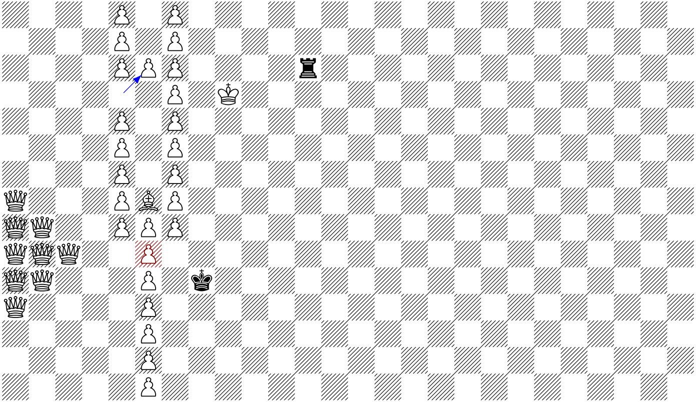
Black to move.

Releasing the Hordes, with value ω^2



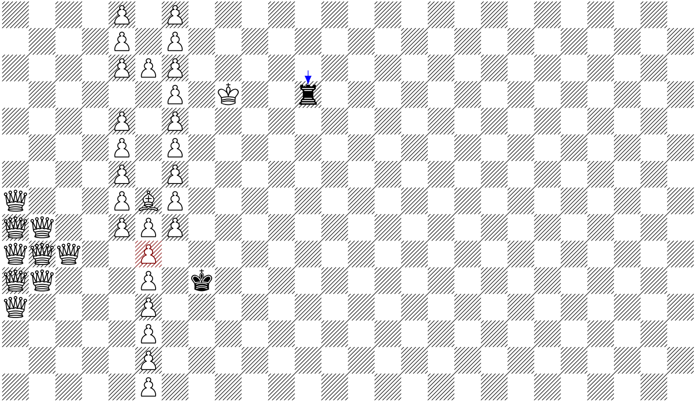
He moves trapped rook up arbitrary height.

Releasing the Hordes, with value ω^2



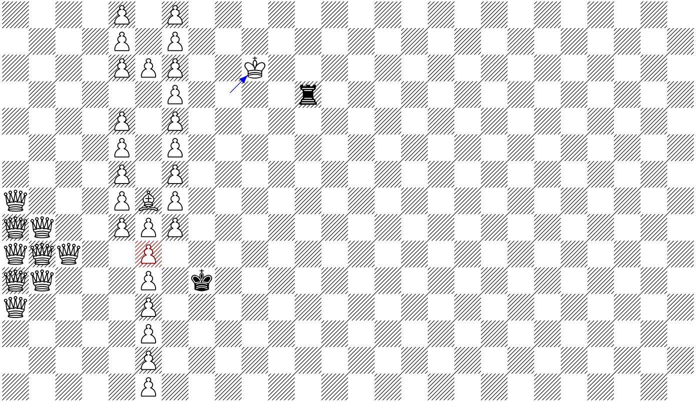
White should capture from left side.

Releasing the Hordes, with value ω^2



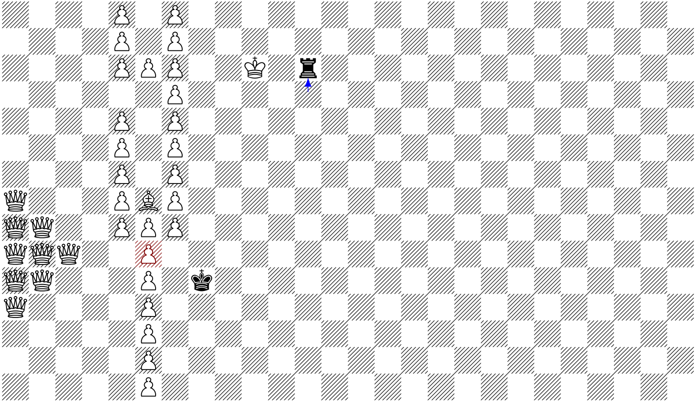
Now black begins to harass white king.

Releasing the Hordes, with value ω^2



White must chase down the rook to avoid perpetual check.

Releasing the Hordes, with value ω^2



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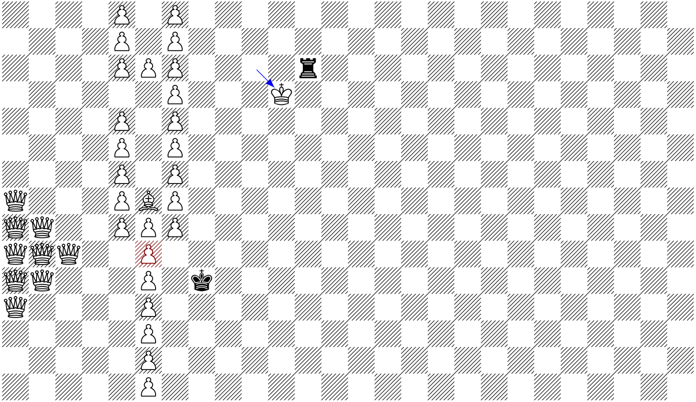

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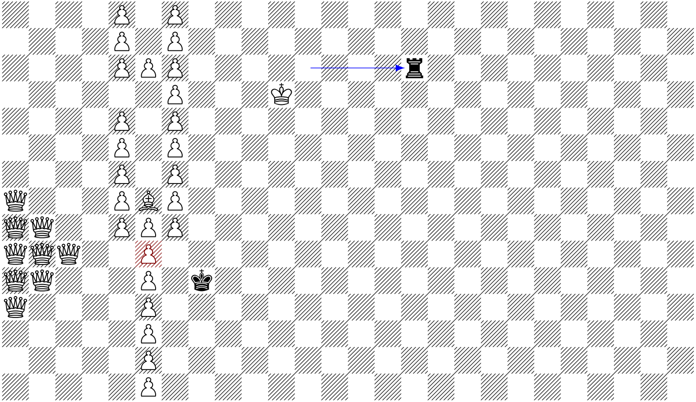
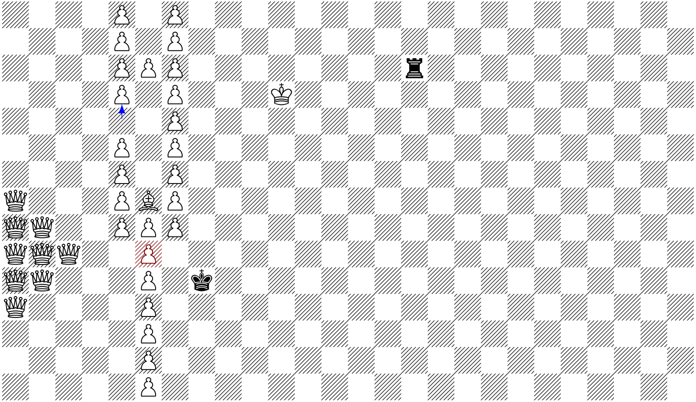


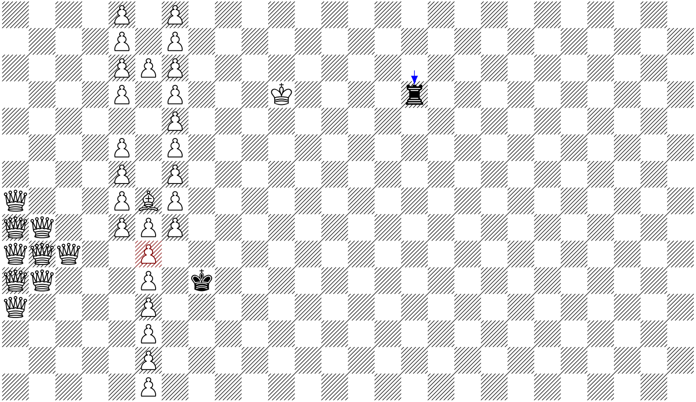
Black must move away to save rook.

Releasing the Hordes, with value ω^2



Now is white's chance to advance a pawn.

Releasing the Hordes, with value ω^2



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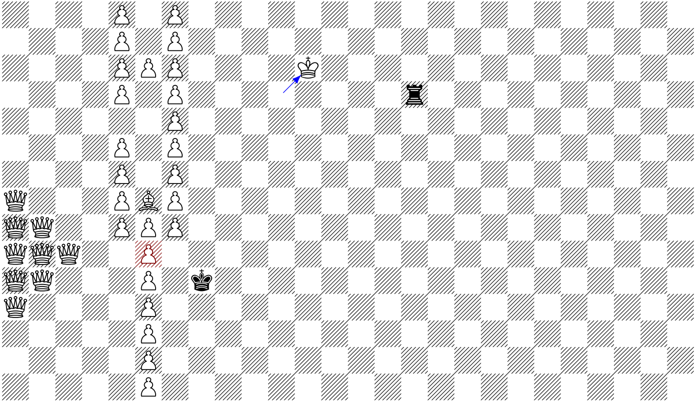

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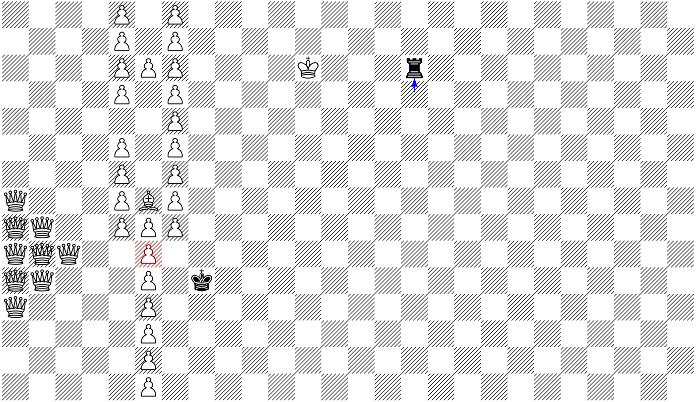

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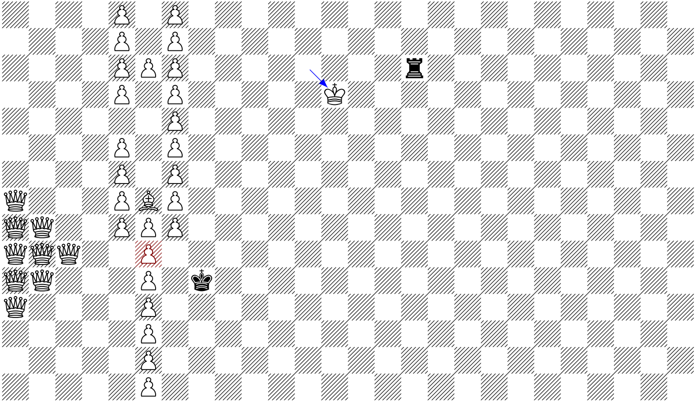

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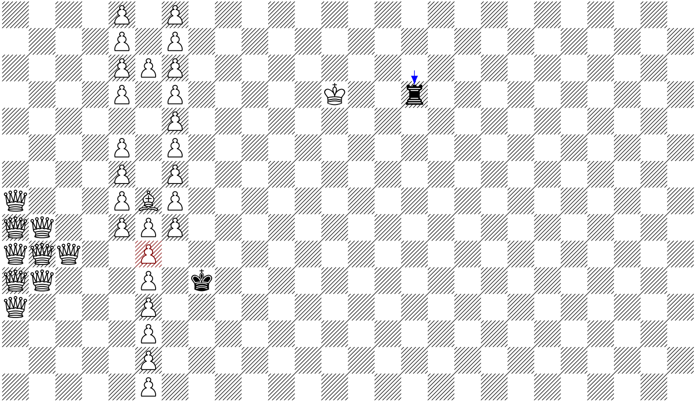

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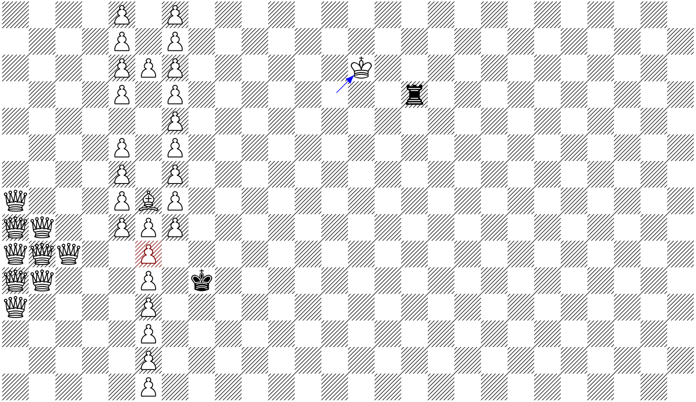

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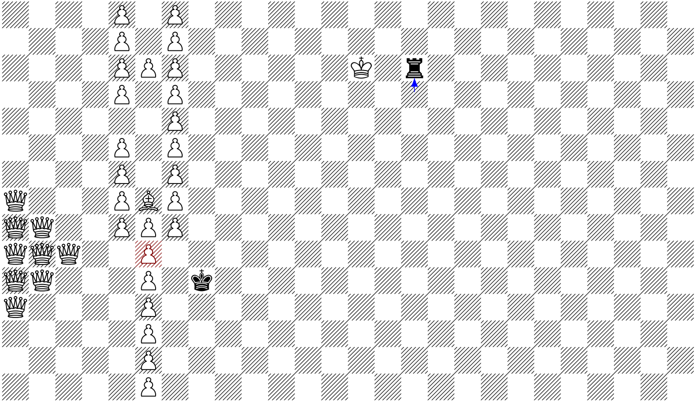

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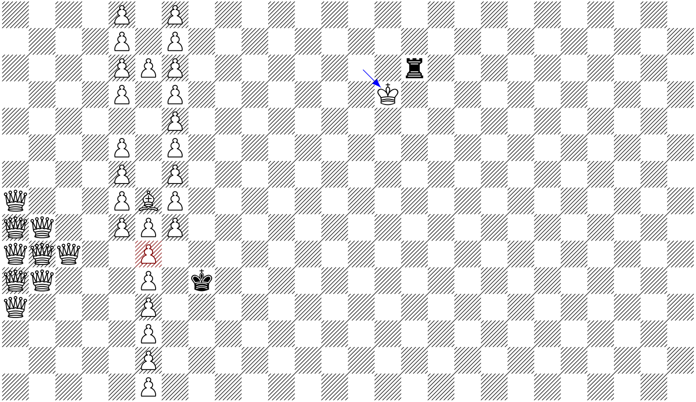

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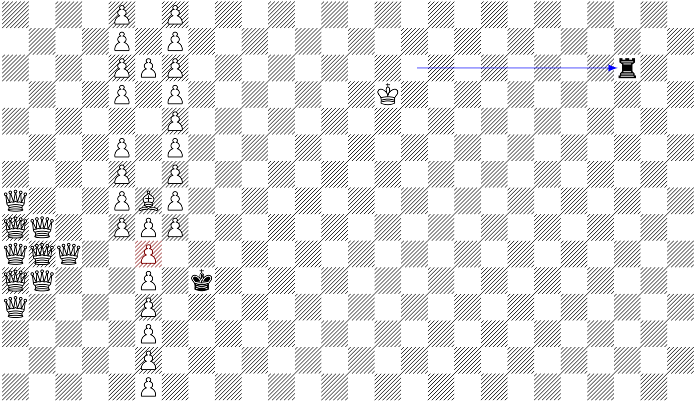
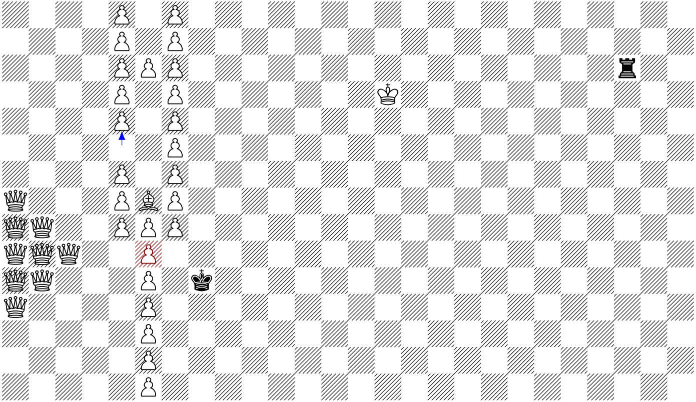


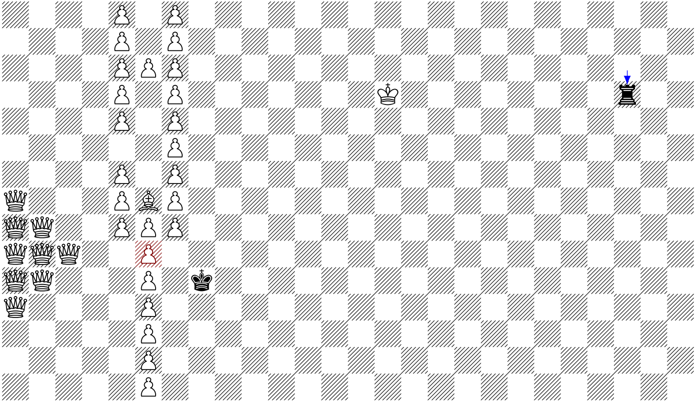
Black moves arbitrary distance out.

Releasing the Hordes, with value ω^2



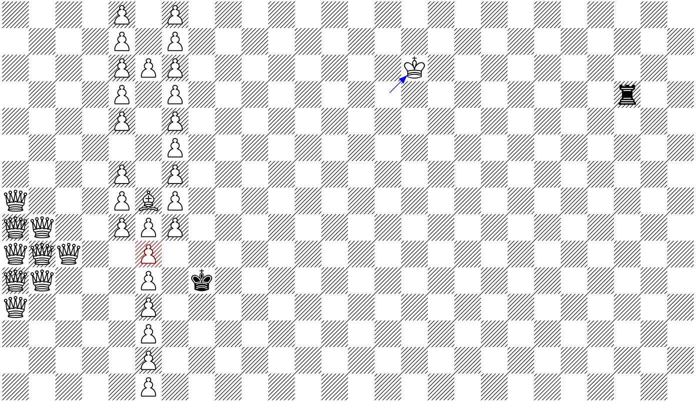
Another chance to advance a pawn.

Releasing the Hordes, with value ω^2



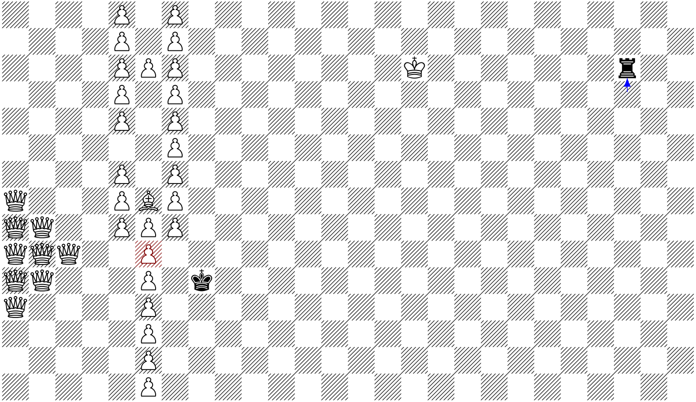
Black harasses the white king.

Releasing the Hordes, with value ω^2

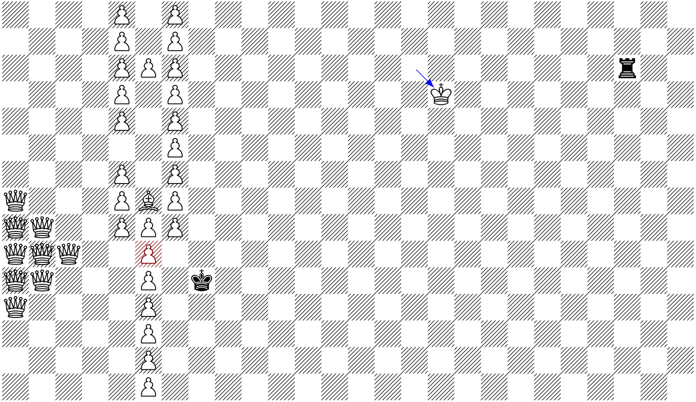


White must chase him down.

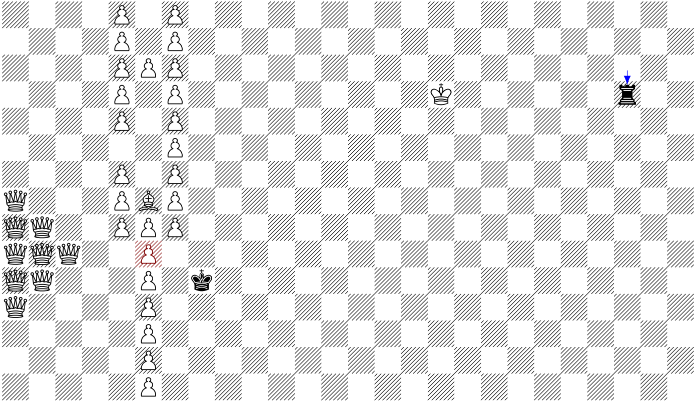
Releasing the Hordes, with value ω^2



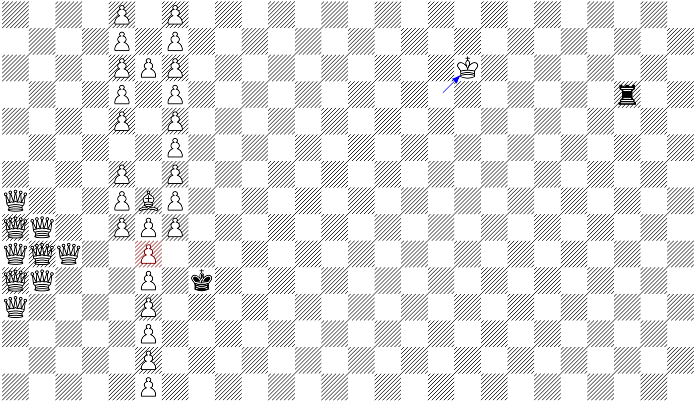
Releasing the Hordes, with value ω^2



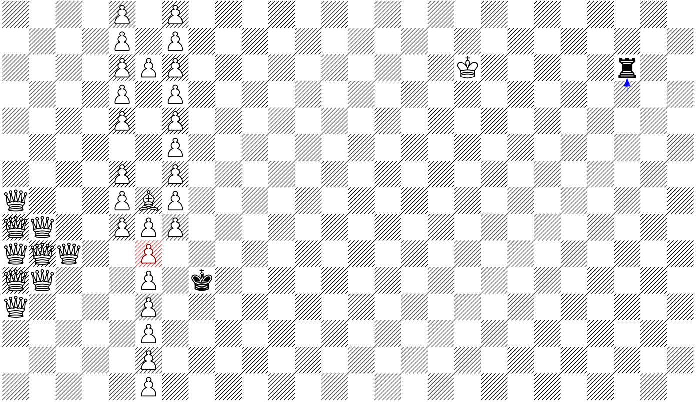
Releasing the Hordes, with value ω^2



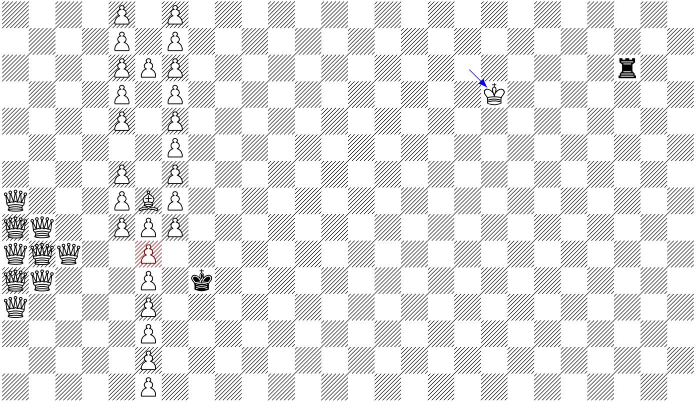
Releasing the Hordes, with value ω^2



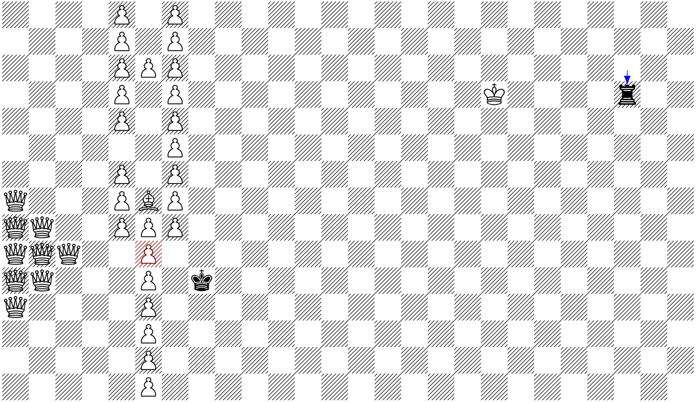
Releasing the Hordes, with value ω^2



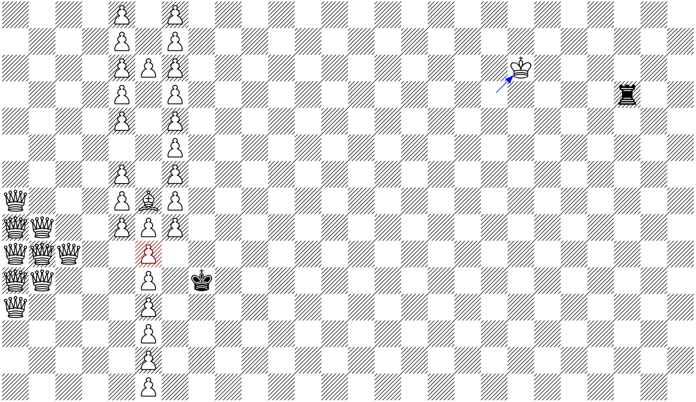
Releasing the Hordes, with value ω^2



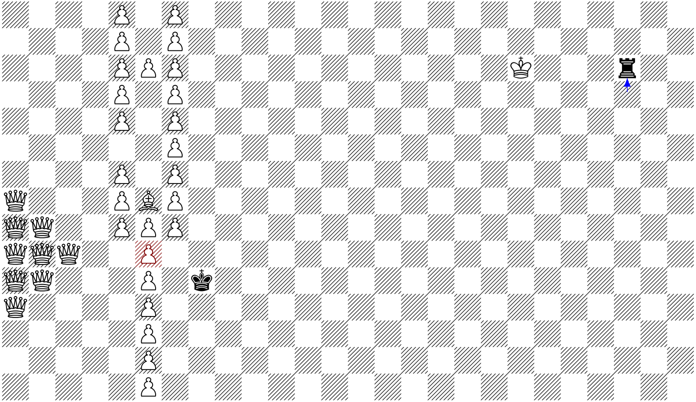
Releasing the Hordes, with value ω^2



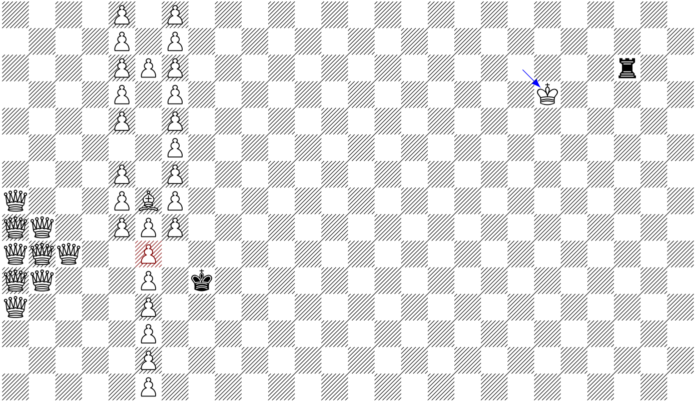
Releasing the Hordes, with value ω^2



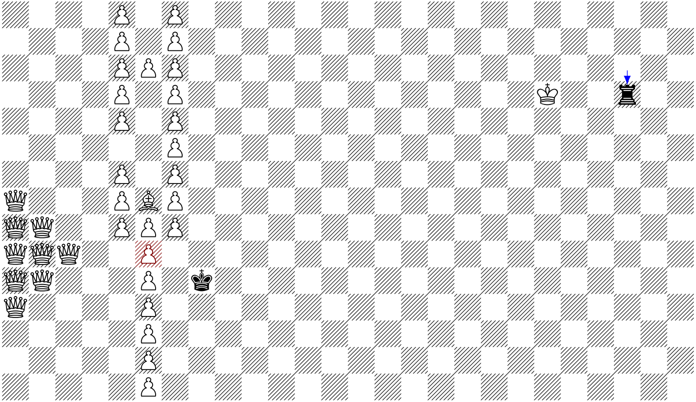
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

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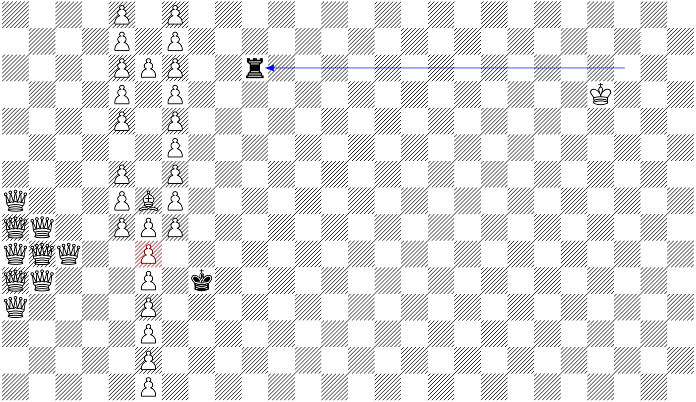
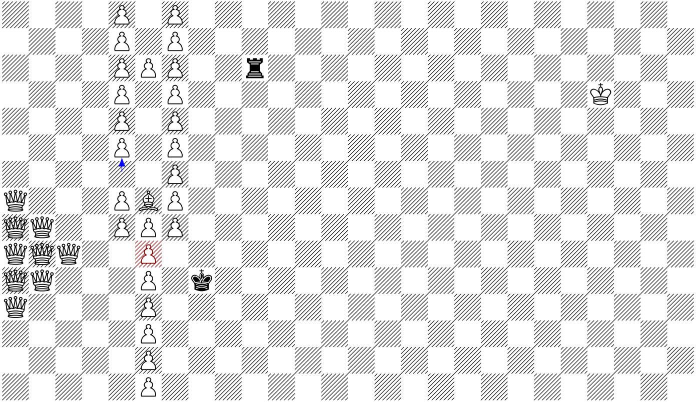
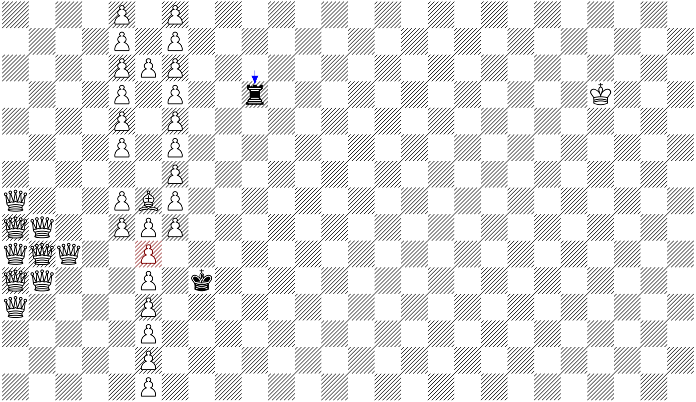


(Black should actually move arbitrary distance to the right.)

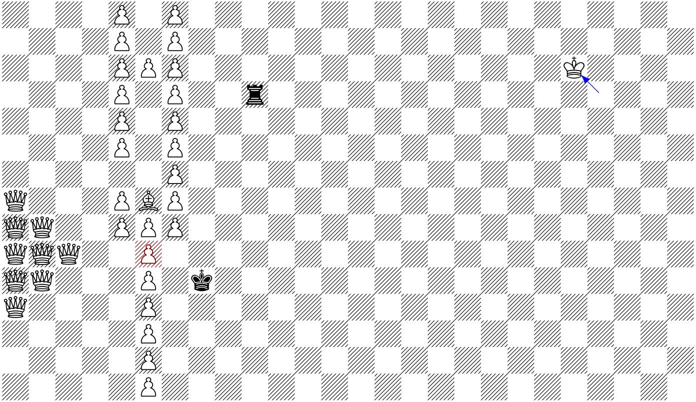
Releasing the Hordes, with value ω^2



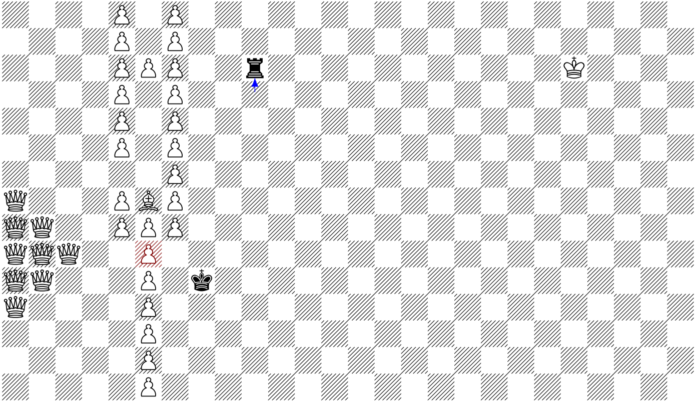
Releasing the Hordes, with value ω^2



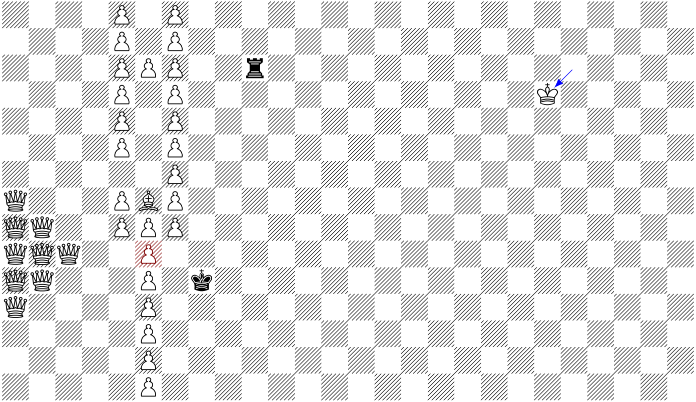
Releasing the Hordes, with value ω^2



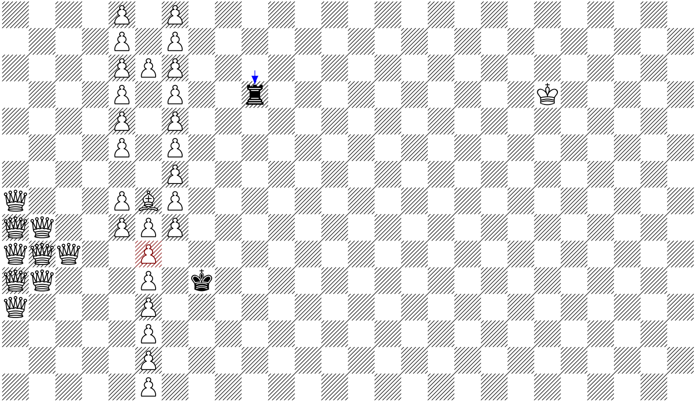
Releasing the Hordes, with value ω^2



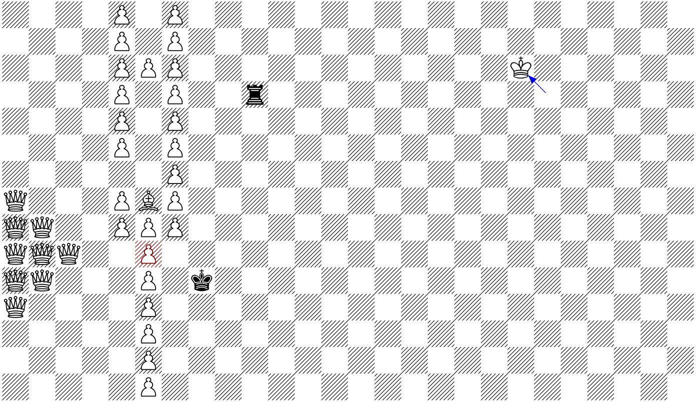
Releasing the Hordes, with value ω^2



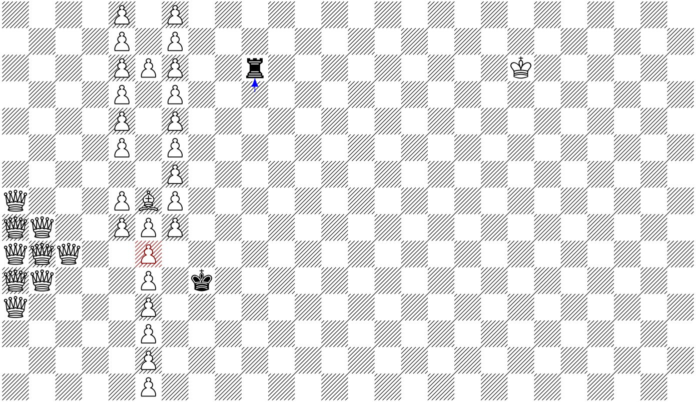
Releasing the Hordes, with value ω^2



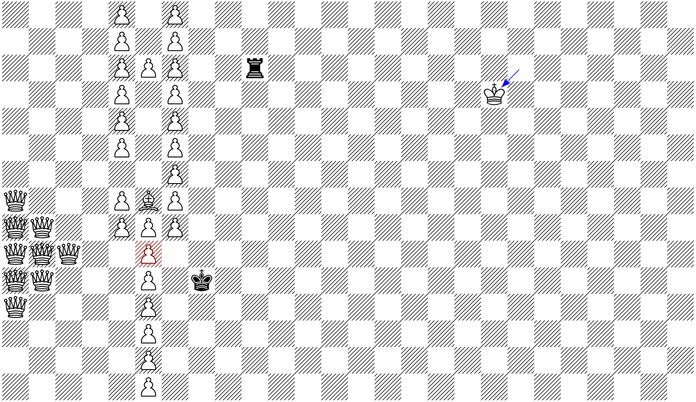
Releasing the Hordes, with value ω^2



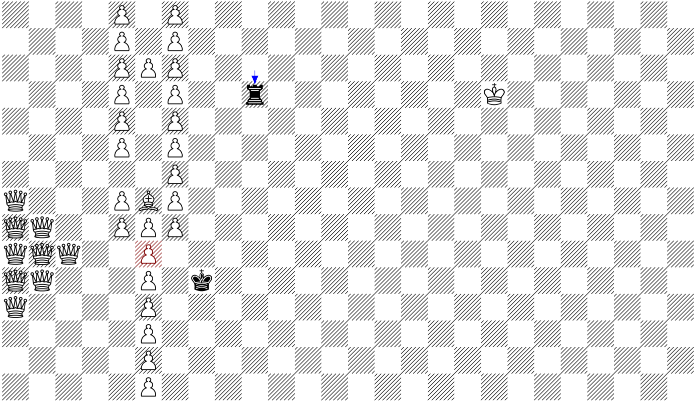
Releasing the Hordes, with value ω^2



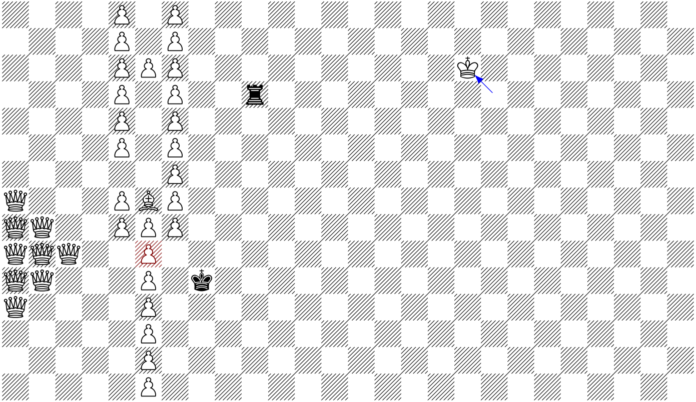
Releasing the Hordes, with value ω^2



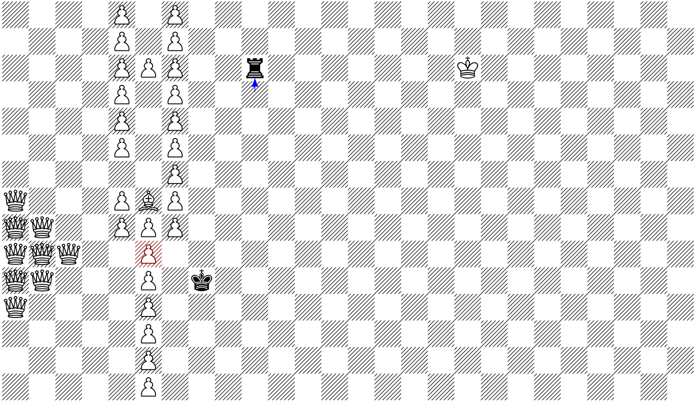
Releasing the Hordes, with value ω^2



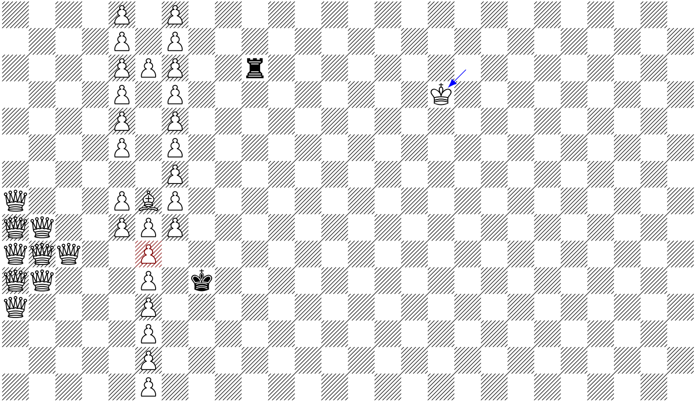
Releasing the Hordes, with value ω^2



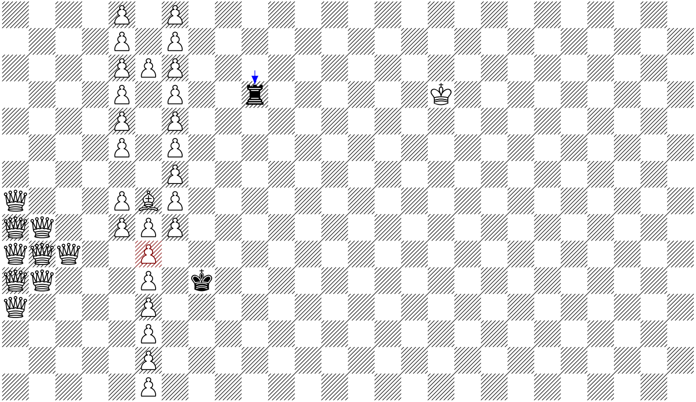
Releasing the Hordes, with value ω^2



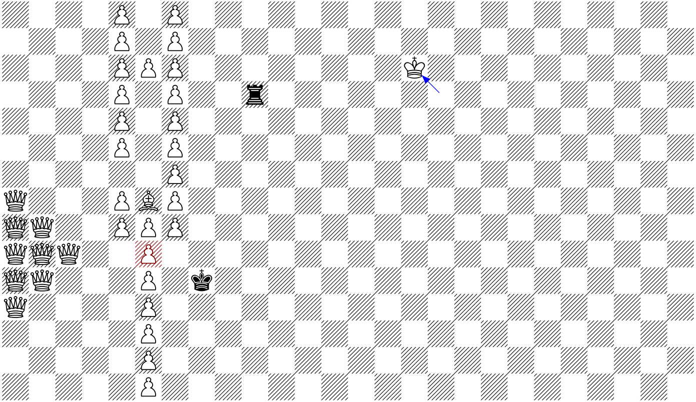
Releasing the Hordes, with value ω^2



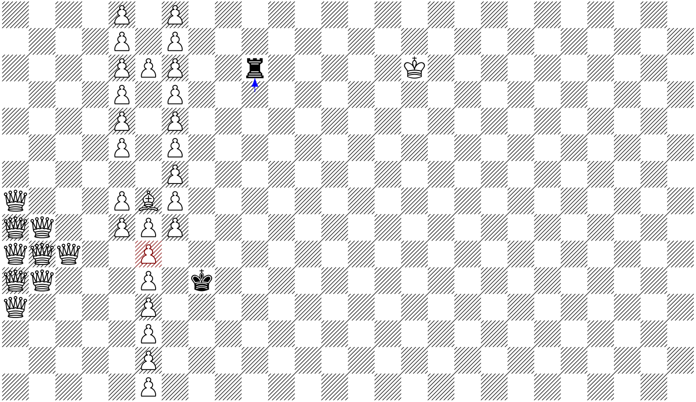
Releasing the Hordes, with value ω^2



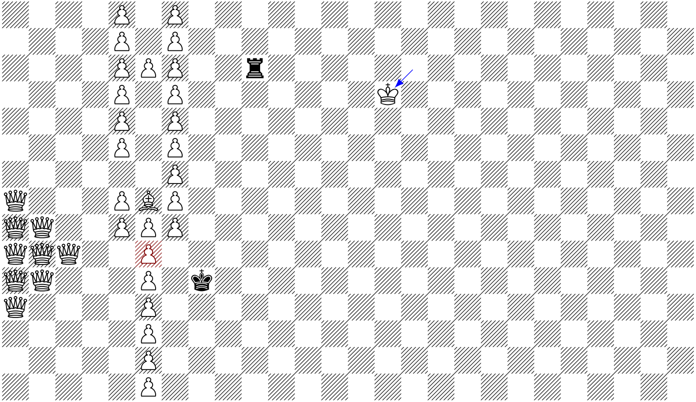
Releasing the Hordes, with value ω^2



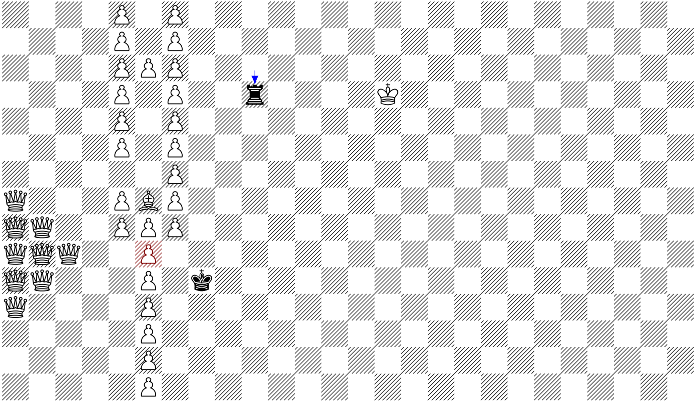
Releasing the Hordes, with value ω^2



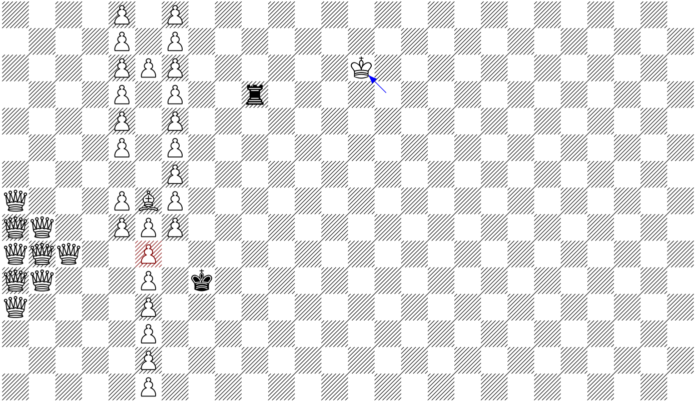
Releasing the Hordes, with value ω^2



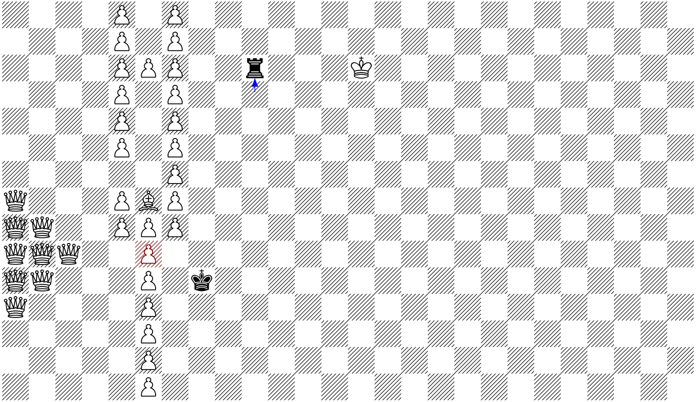
Releasing the Hordes, with value ω^2



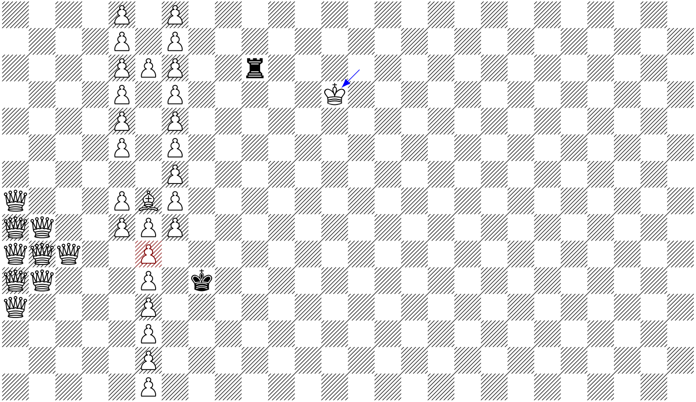
Releasing the Hordes, with value ω^2



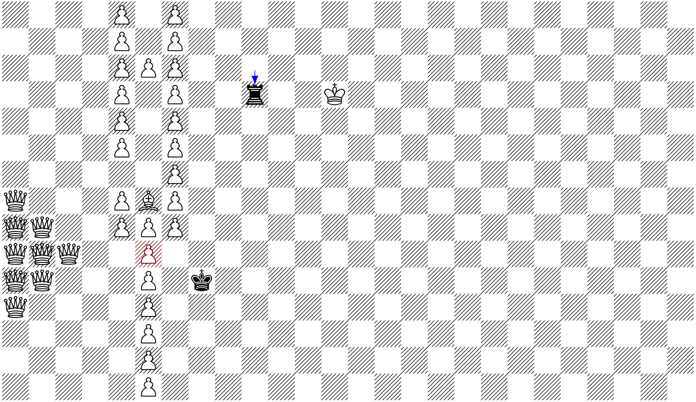
Releasing the Hordes, with value ω^2



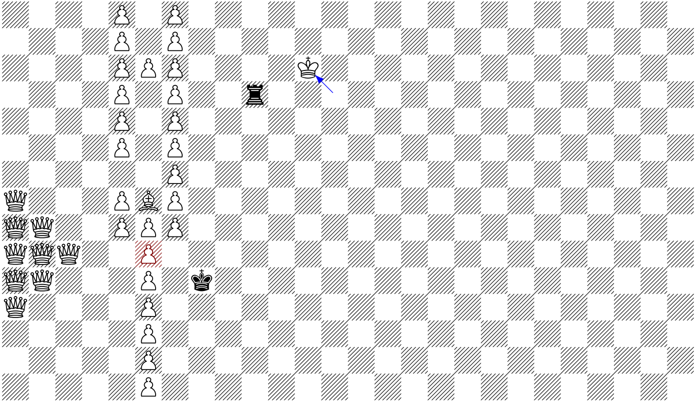
Releasing the Hordes, with value ω^2



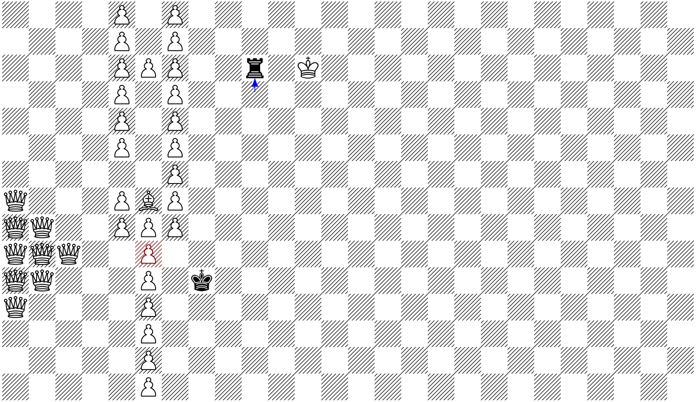
Releasing the Hordes, with value ω^2



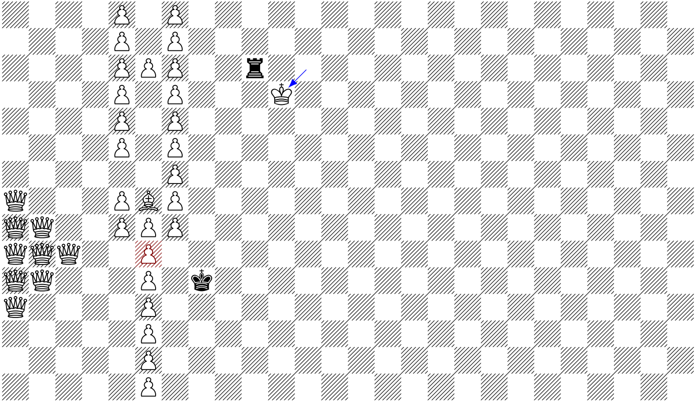
Releasing the Hordes, with value ω^2



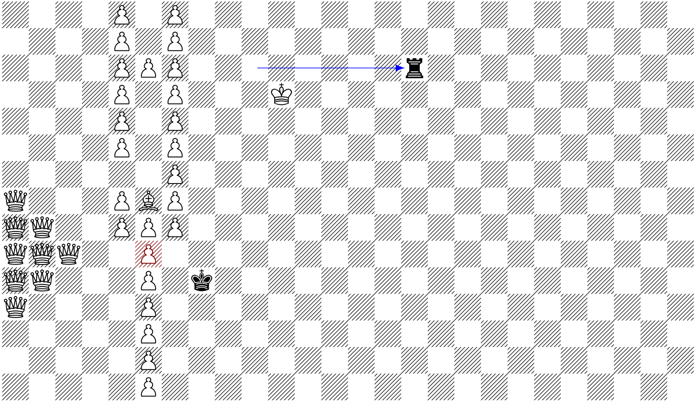
Releasing the Hordes, with value ω^2



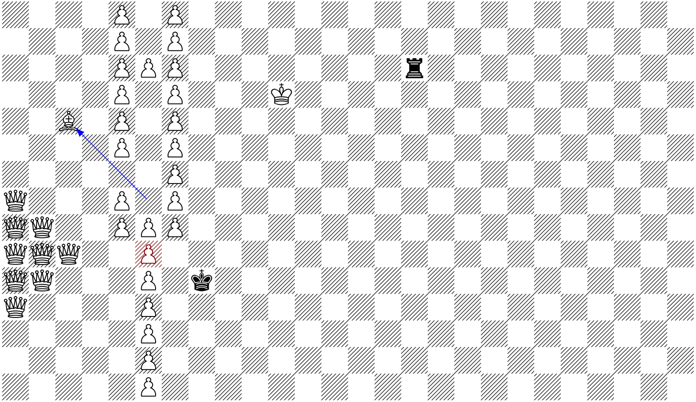
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

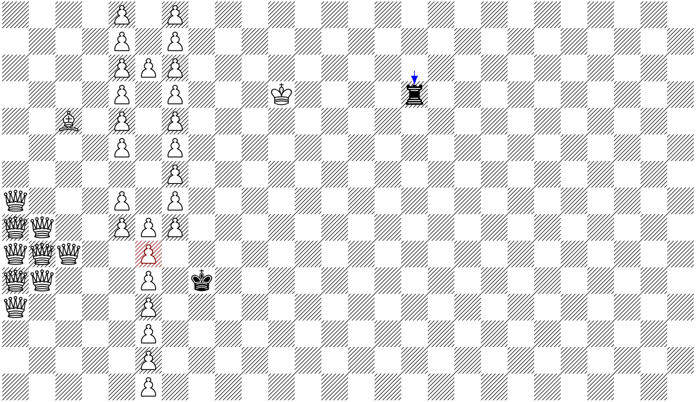


Releasing the Hordes, with value ω^2

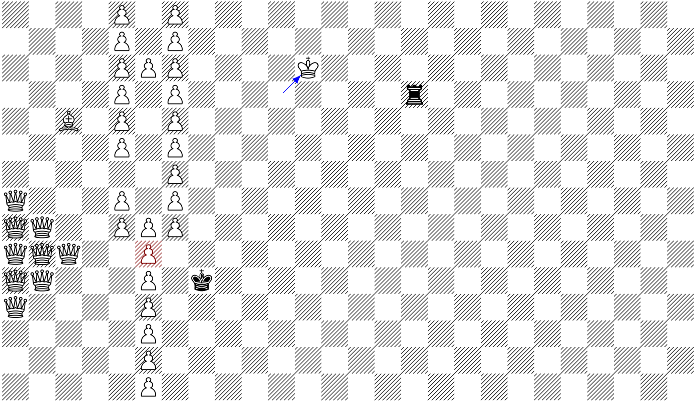


The bishop unlocks the door.

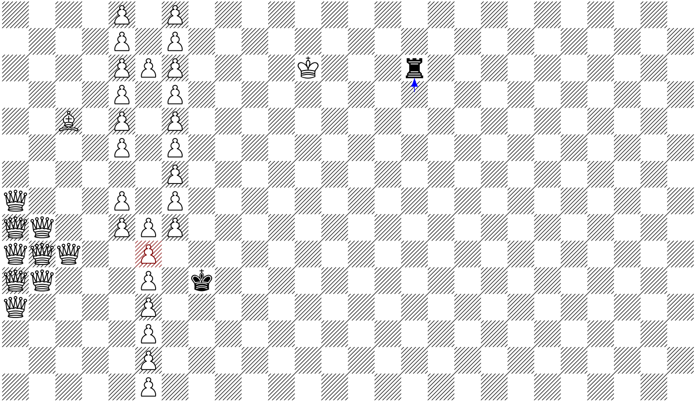
Releasing the Hordes, with value ω^2



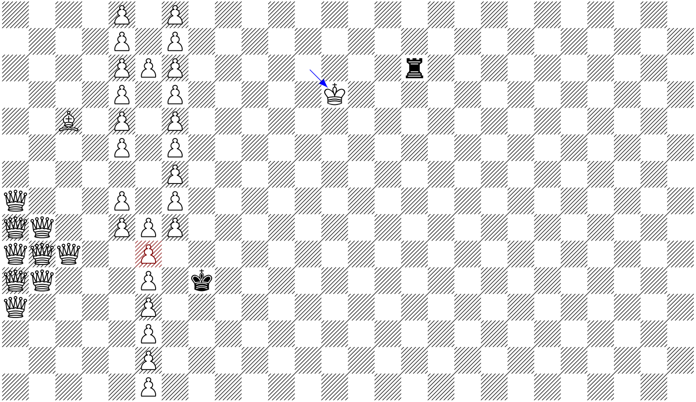
Releasing the Hordes, with value ω^2



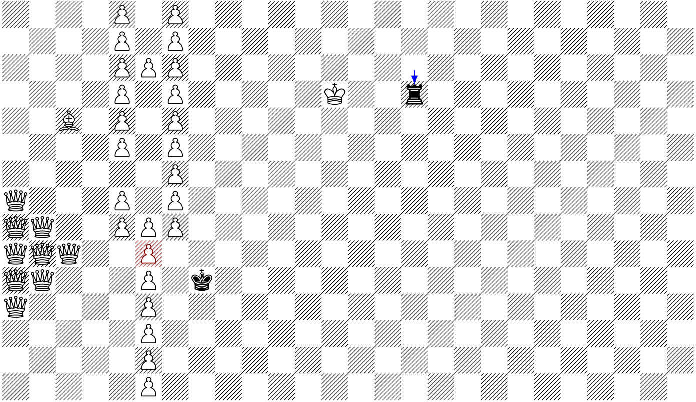
Releasing the Hordes, with value ω^2



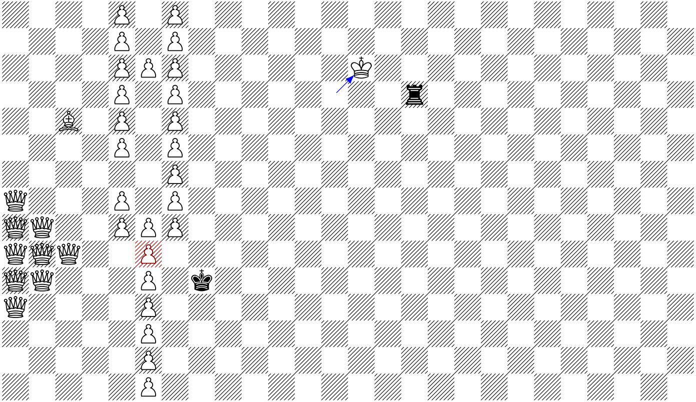
Releasing the Hordes, with value ω^2



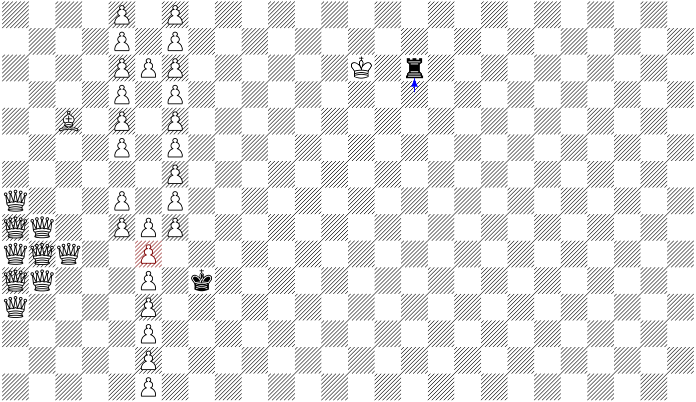
Releasing the Hordes, with value ω^2



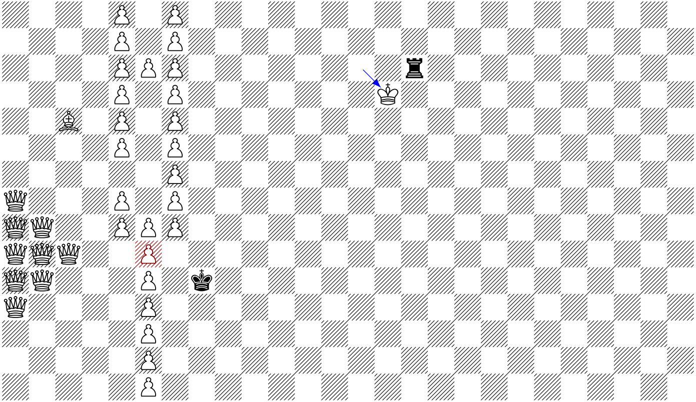
Releasing the Hordes, with value ω^2



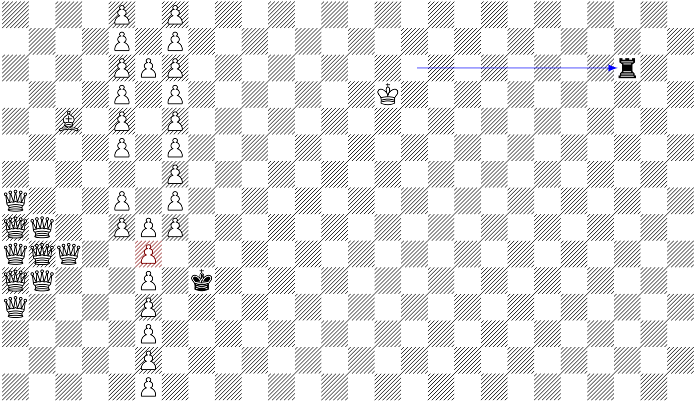
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

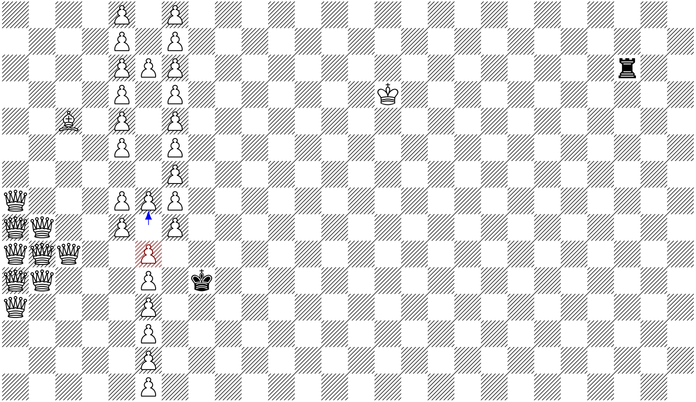


Releasing the Hordes, with value ω^2

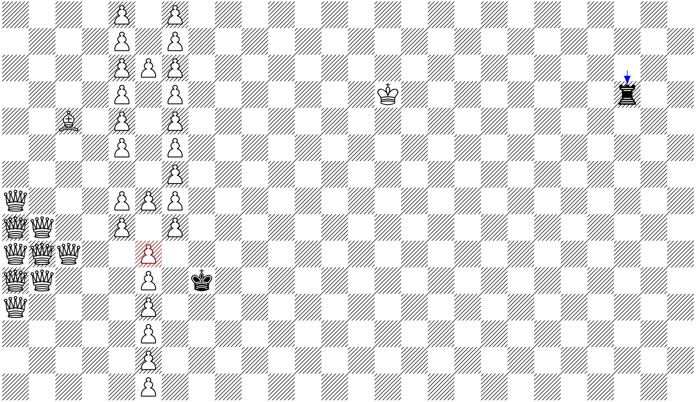


Black can move rook arbitrary distance.

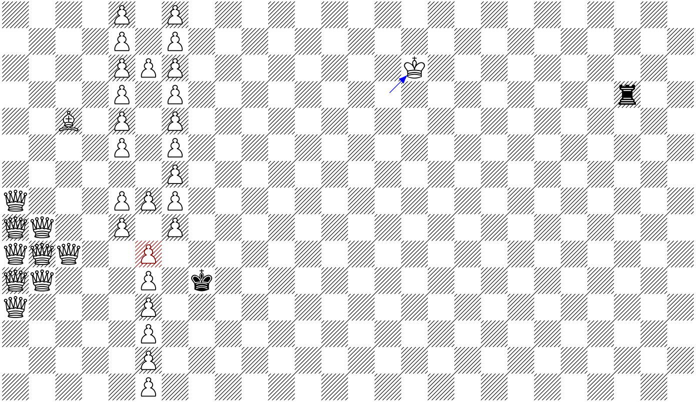
Releasing the Hordes, with value ω^2



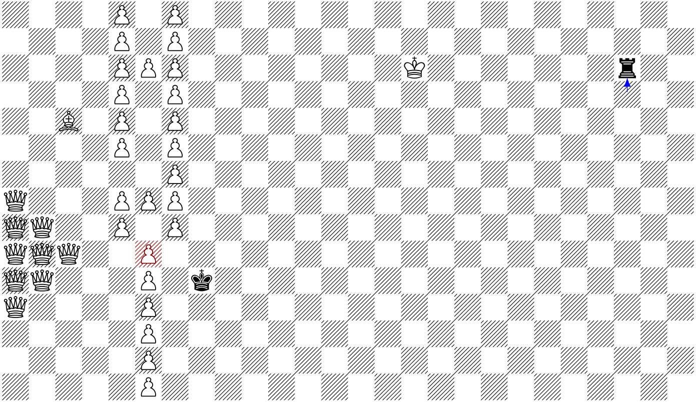
Releasing the Hordes, with value ω^2



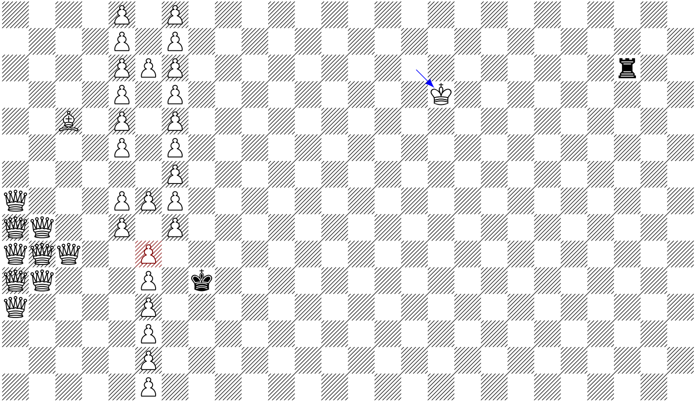
Releasing the Hordes, with value ω^2



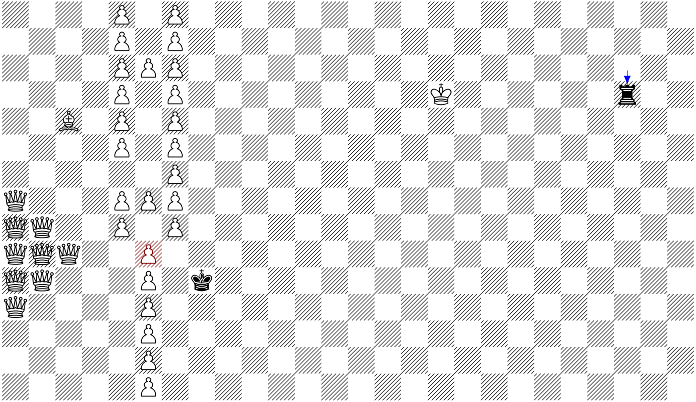
Releasing the Hordes, with value ω^2



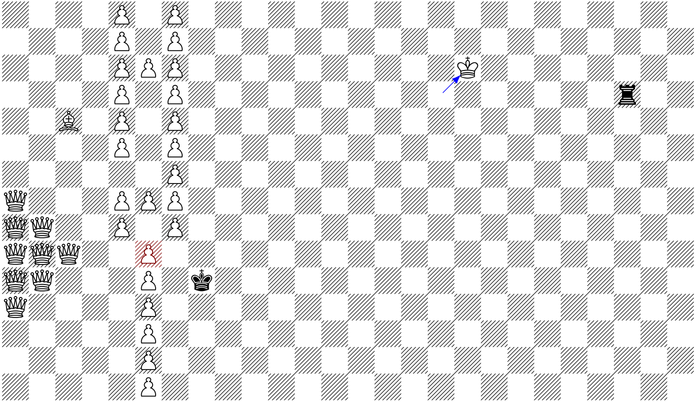
Releasing the Hordes, with value ω^2



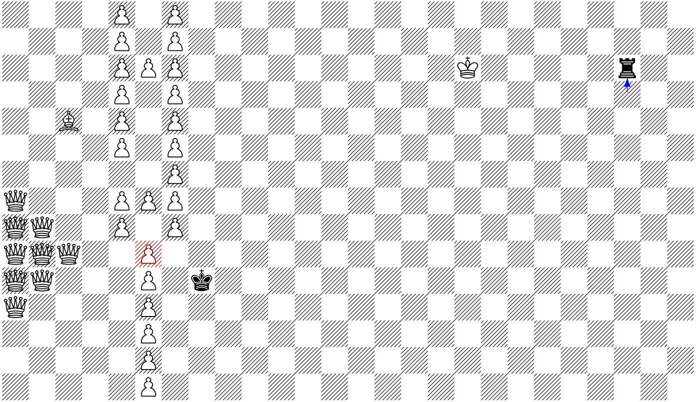
Releasing the Hordes, with value ω^2



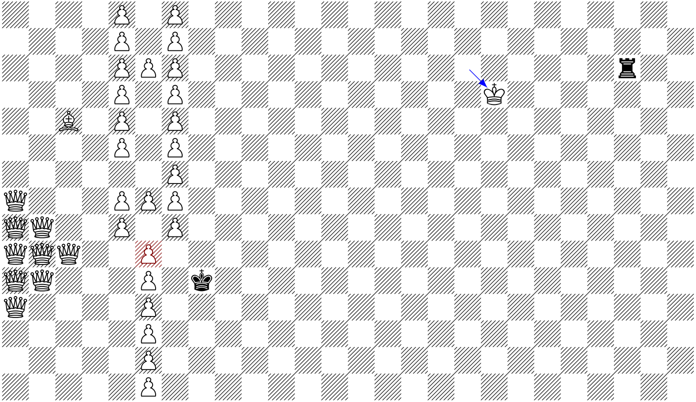
Releasing the Hordes, with value ω^2



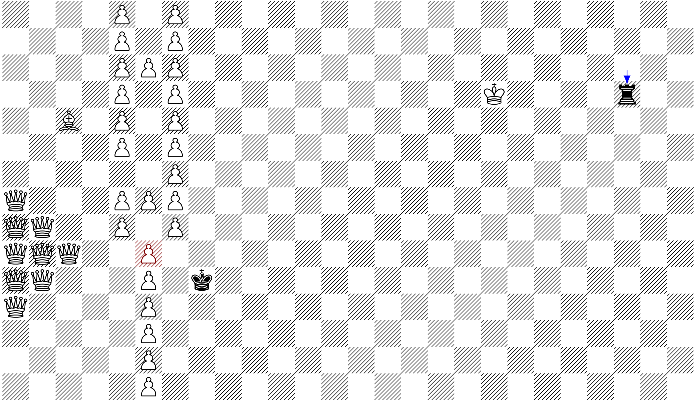
Releasing the Hordes, with value ω^2



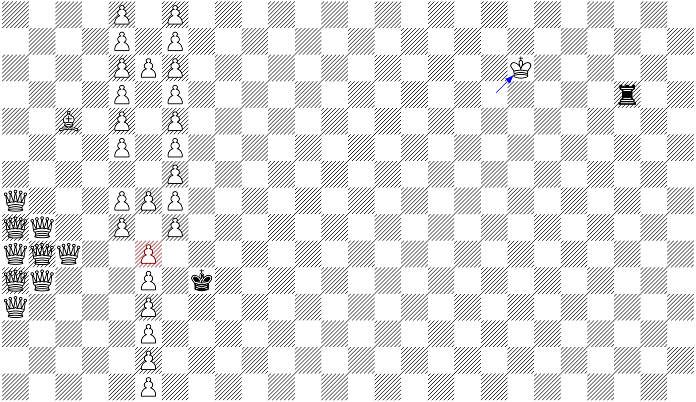
Releasing the Hordes, with value ω^2



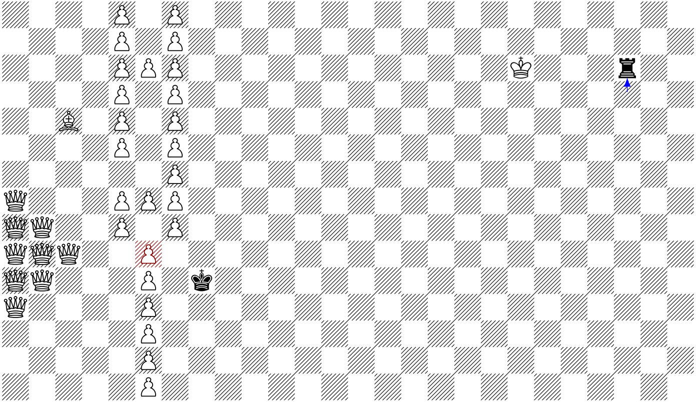
Releasing the Hordes, with value ω^2



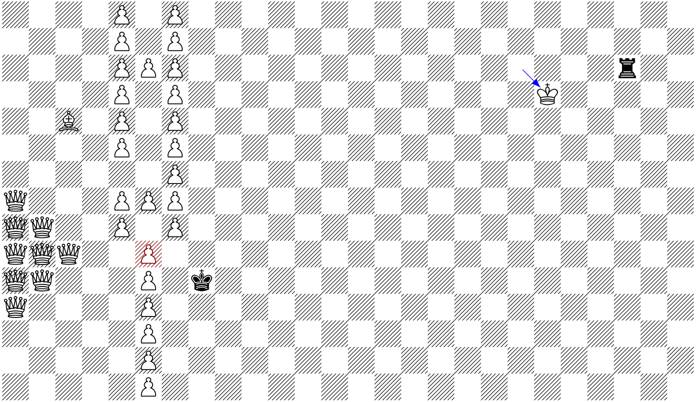
Releasing the Hordes, with value ω^2



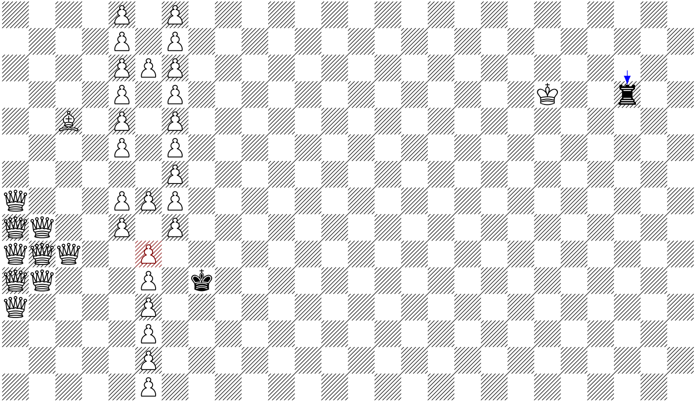
Releasing the Hordes, with value ω^2



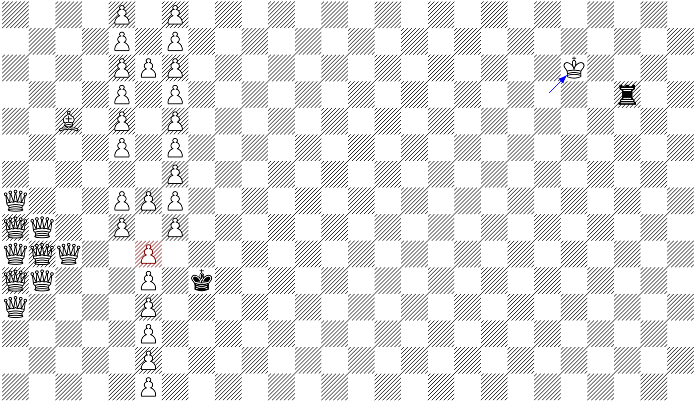
Releasing the Hordes, with value ω^2



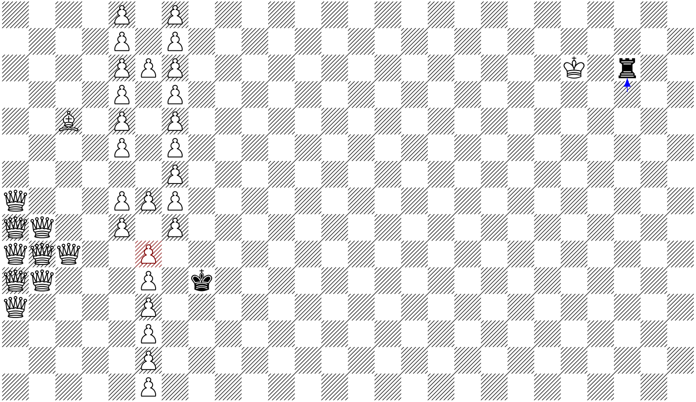
Releasing the Hordes, with value ω^2



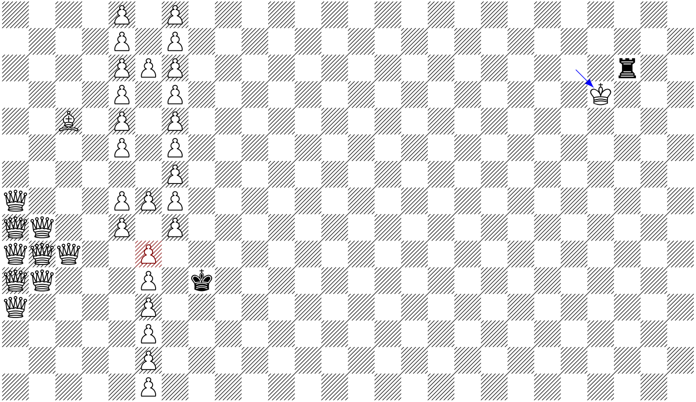
Releasing the Hordes, with value ω^2



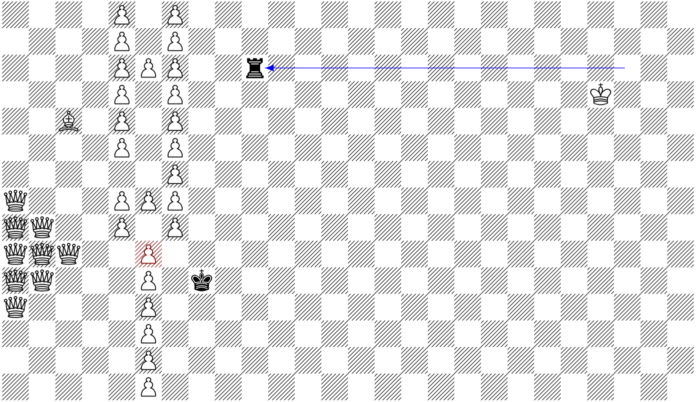
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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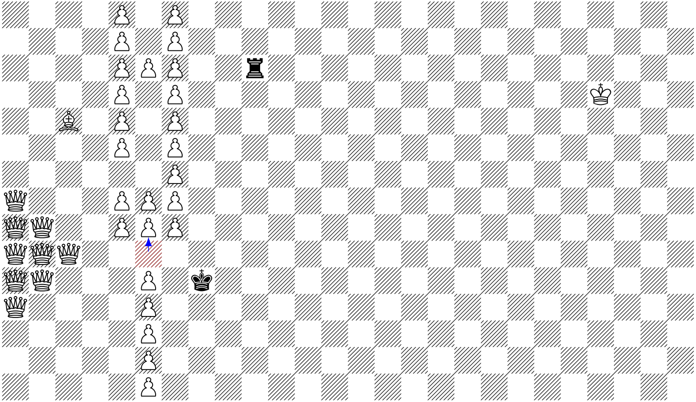
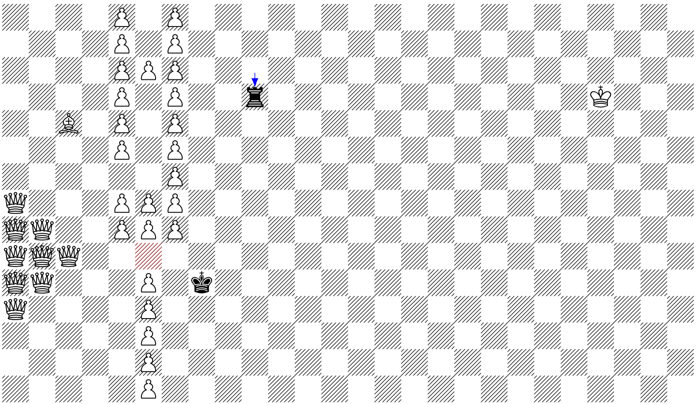


The portcullis opens...

Releasing the Hordes, with value ω^2



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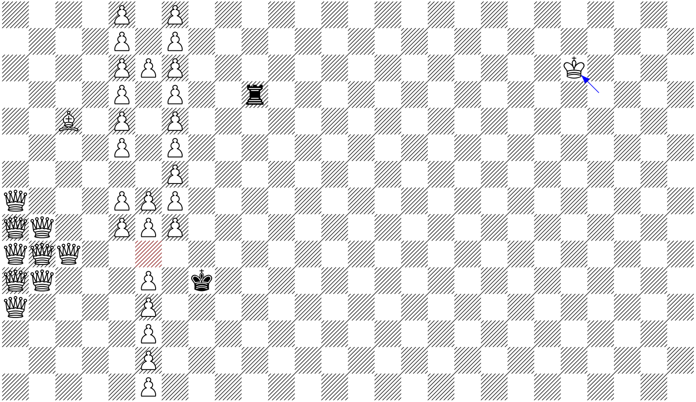

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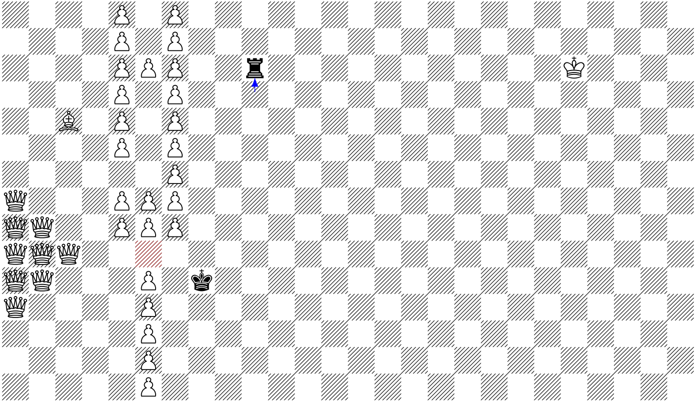

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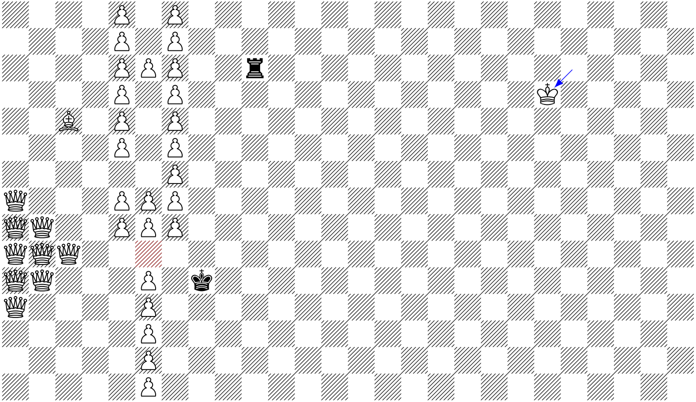

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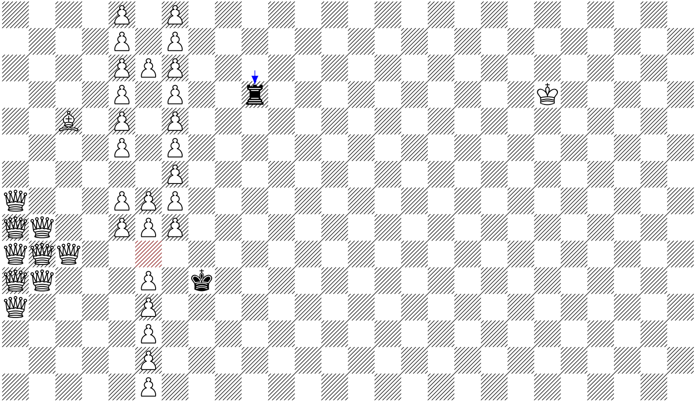

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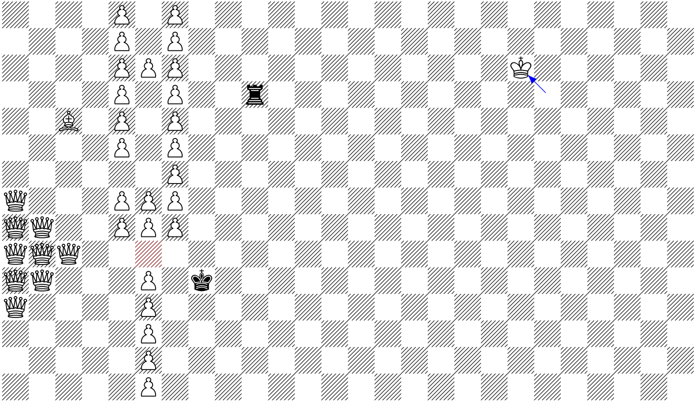

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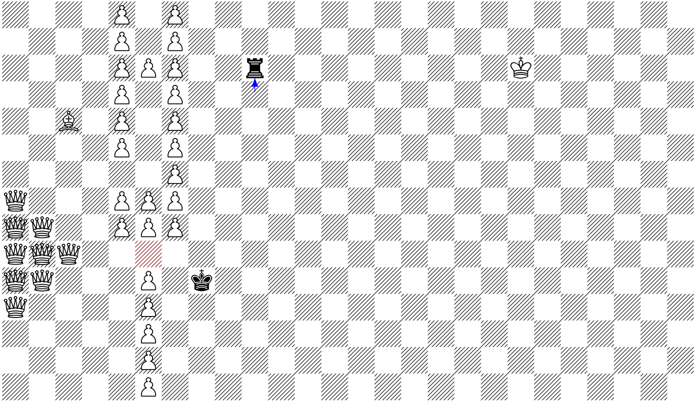

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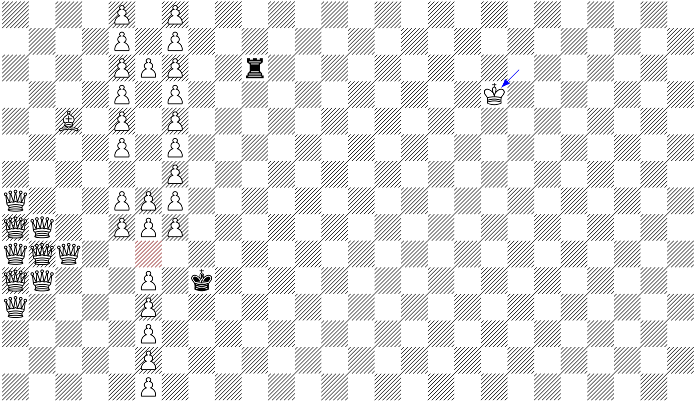

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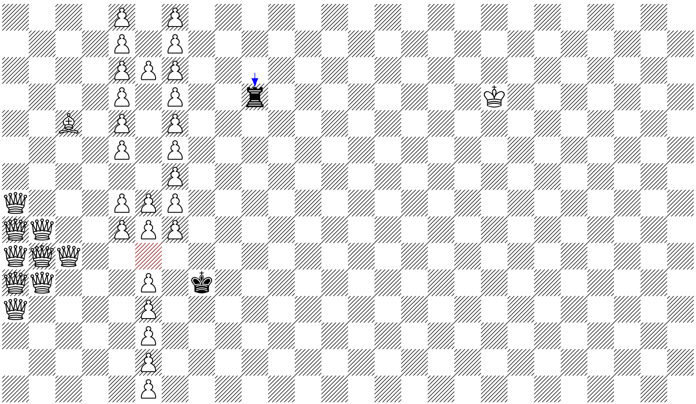

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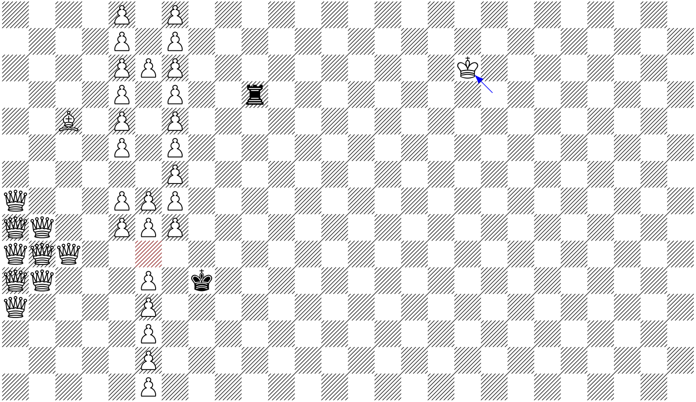

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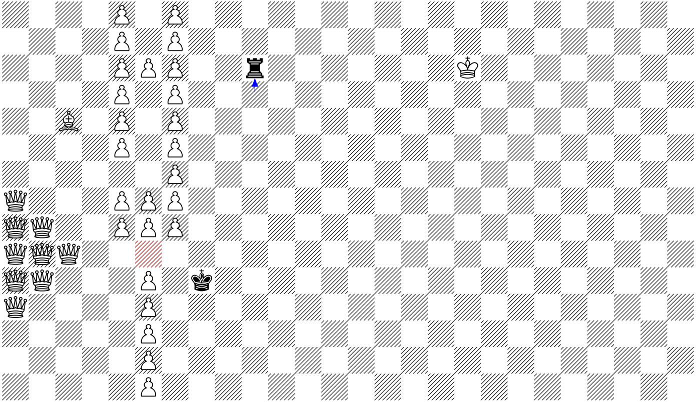

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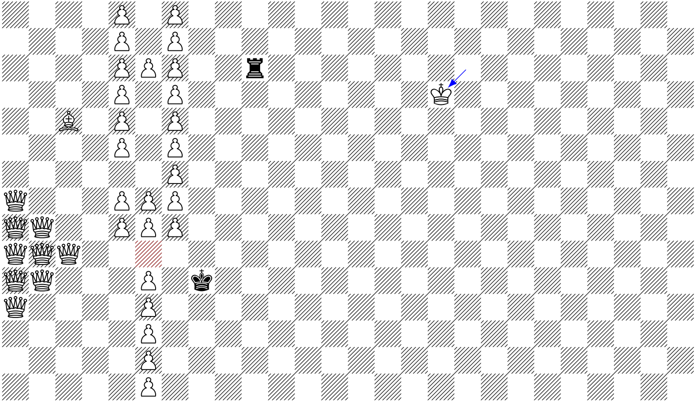

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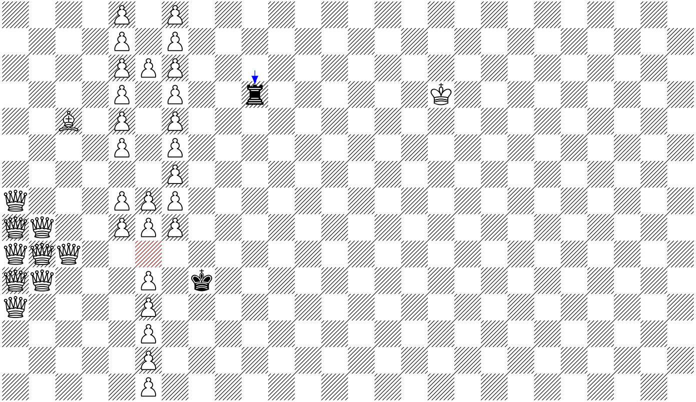

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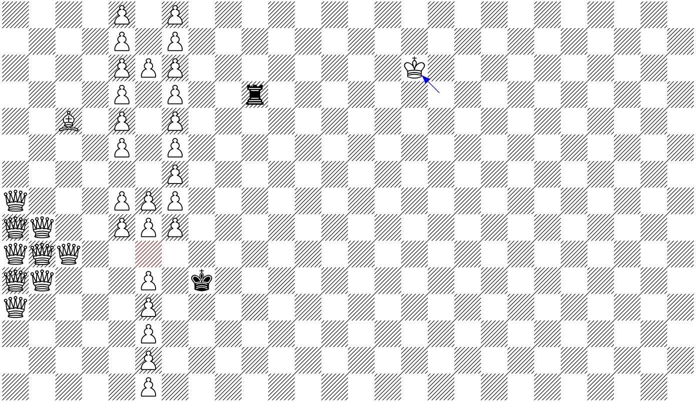

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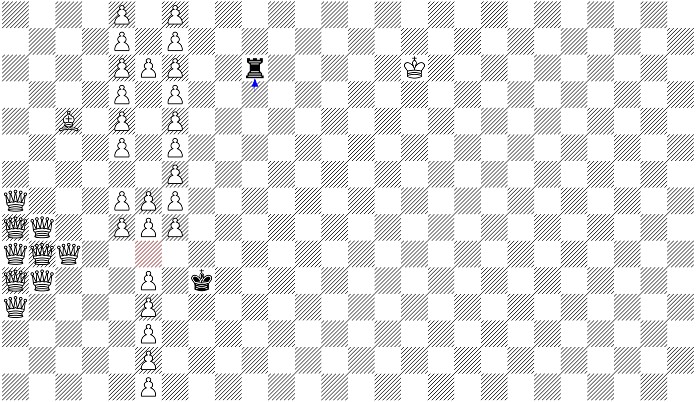

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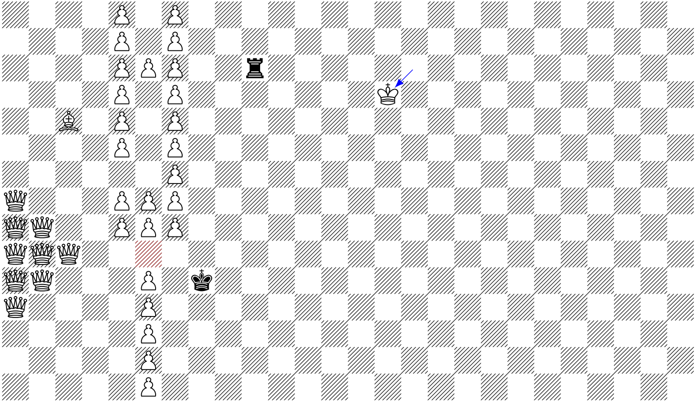

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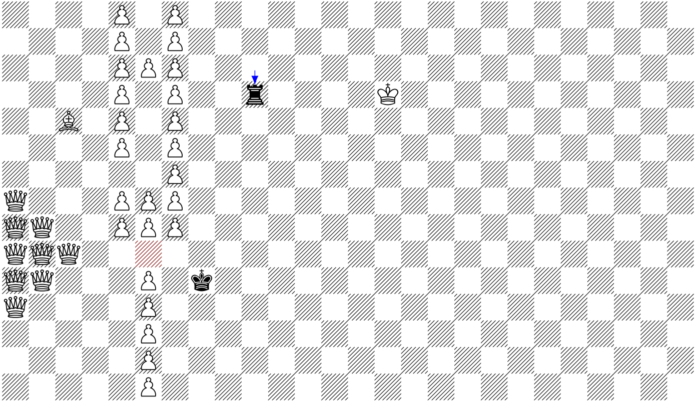

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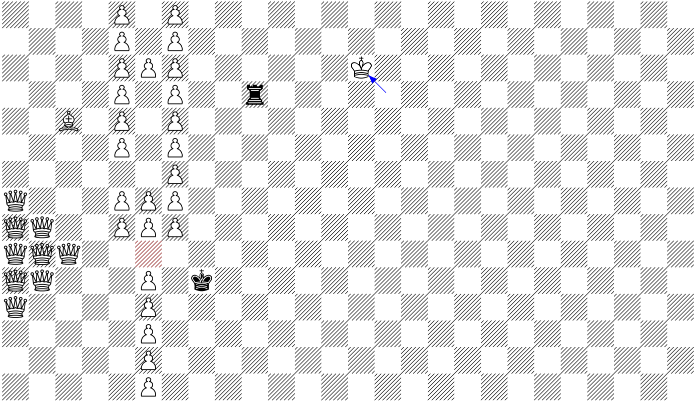

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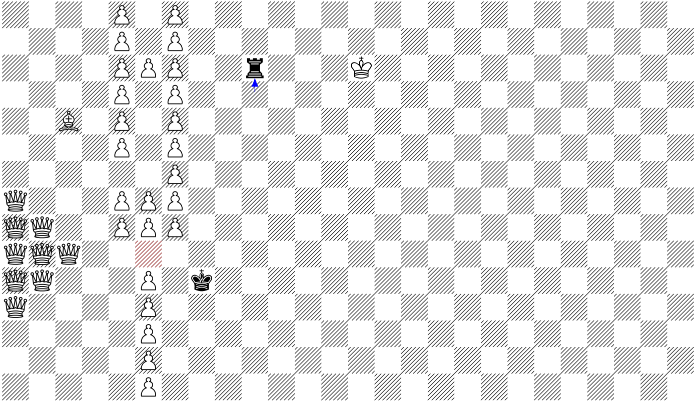

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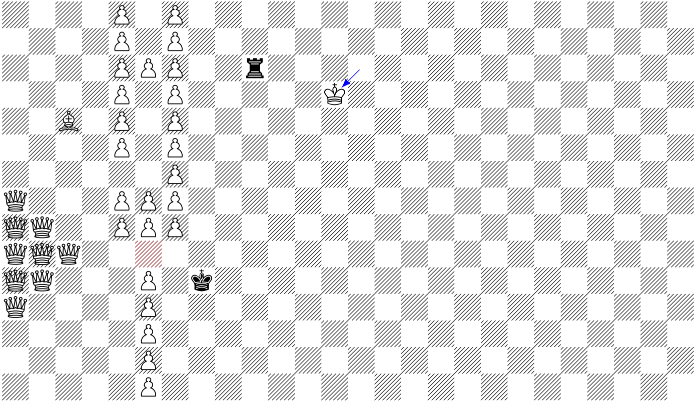

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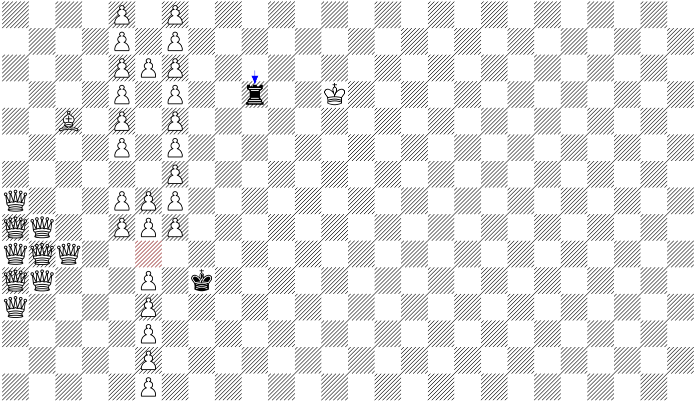

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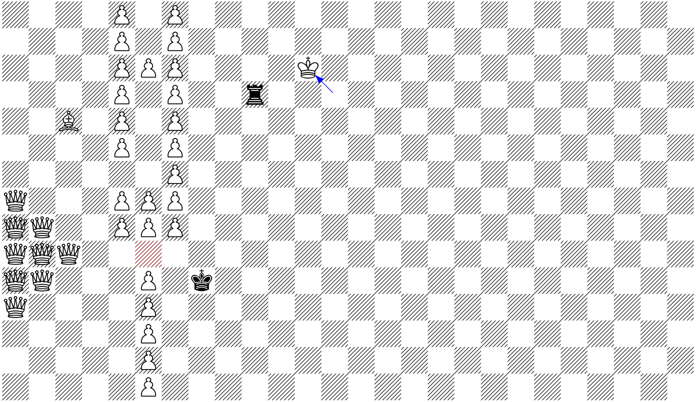

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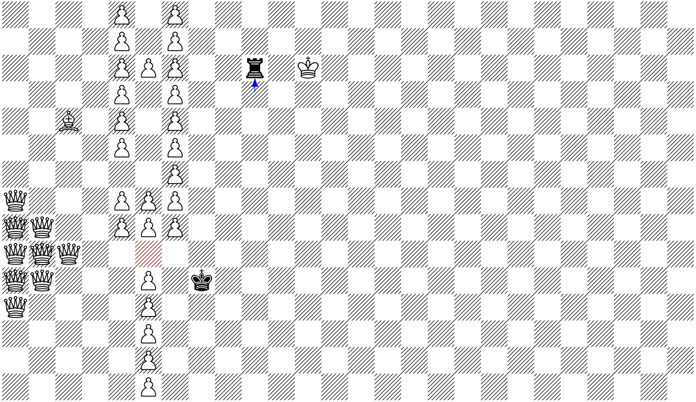

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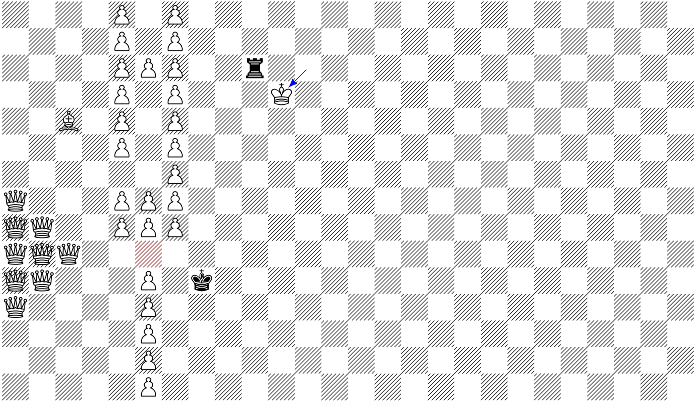

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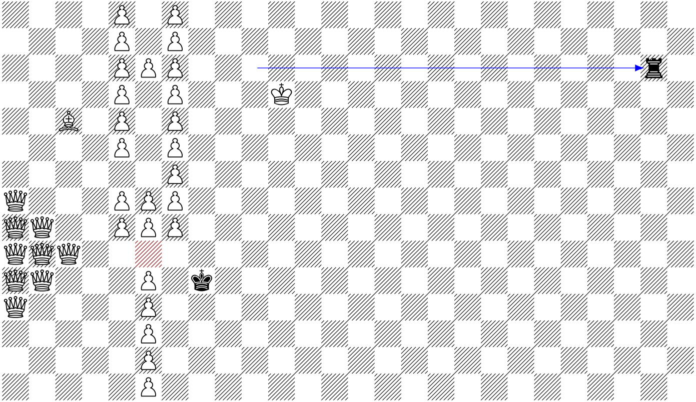

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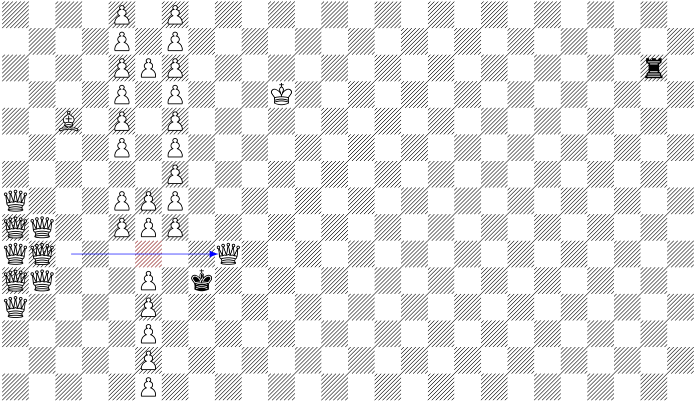
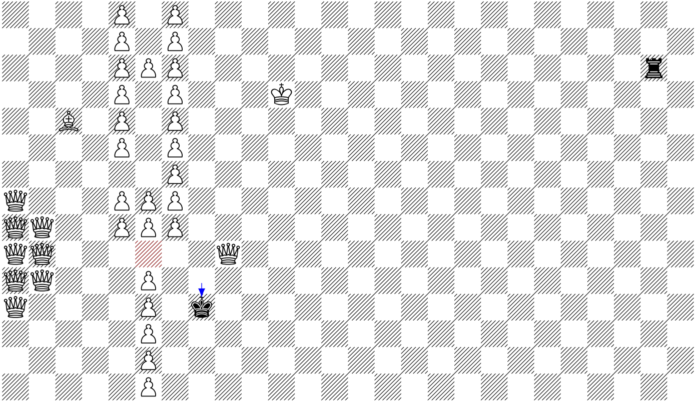
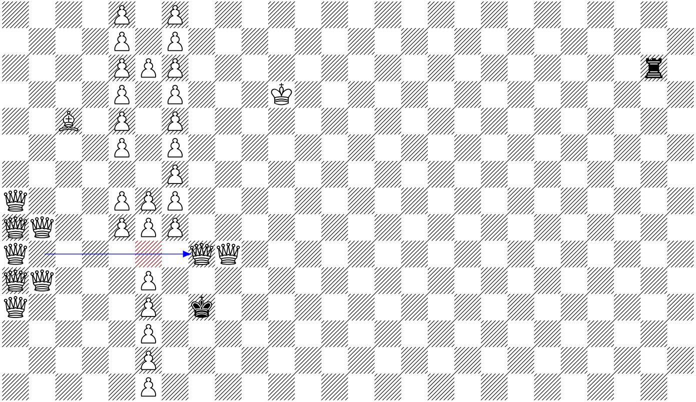


Queens enter the mating chamber.

Releasing the Hordes, with value ω^2

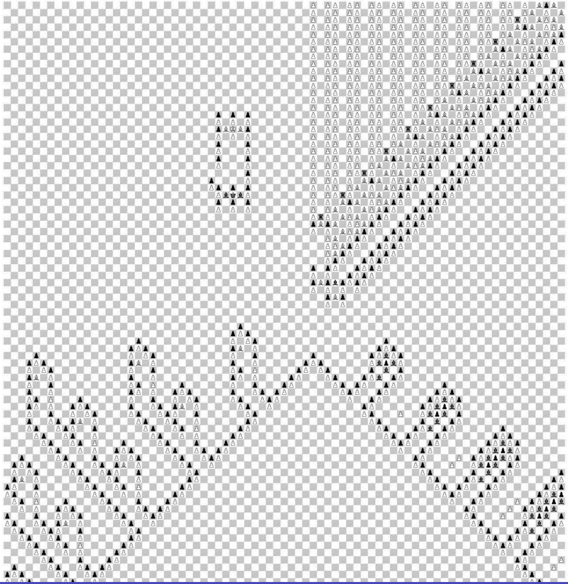


Releasing the Hordes, with value ω^2

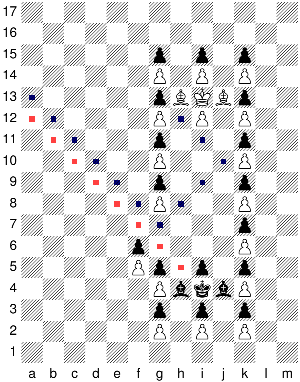


Checkmate

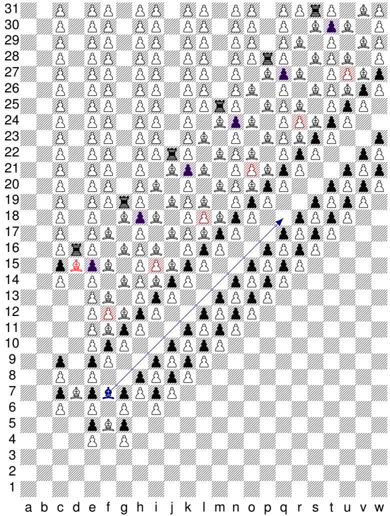
A position with value ω^4



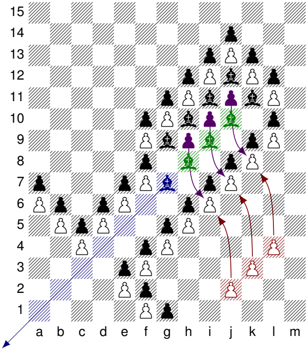
The throne room



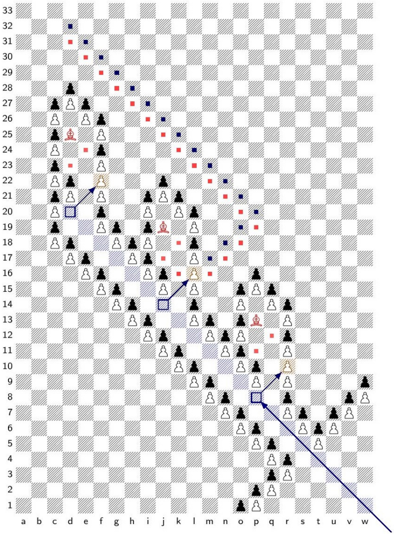
The rook towers



Bishop cannon



Bishop gateway terminal



Infinite chess is universal

Theorem (Mathew Bolan, Andreas Tsevas [BT26])

Every countable ordinal arises as the game value of a position in infinite chess.

The proof method uses the idea of coding well-founded trees into infinite chess.

Theorem (Mathew Bolan, Andreas Tsevas)

Infinite chess is universal, in the sense that every open Gale-Stewart game with draws is strategically equivalent to some infinite chess position and vice versa.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
O'Hara Professor of Logic
University of Notre Dame

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The Midnight Ball

A festive ball, with infinitely many guests.

Everyone has a natural number written on their forehead in black lipstick.

We can see the other numbers, but not our own.

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We can see the other numbers, but not our own.

At midnight, one of us will be selected as Queen of the Ball, allowed to erase her number and have one of her choosing.

Then, we all shout our best predictions for our own numbers.

How well can we do?

We are allowed to plan in advance, but no communication is allowed once the numbers are assigned.

First steps

Of course, the queen can guess correctly.

First steps

Of course, the queen can guess correctly.

Better, she can communicate a finite amount of information with her choice of number.

That is, we can agree on a finite set of us, and let the queen pick the number with prime factorization

$$p_0^{k_0} p_1^{k_1} \cdots p_n^{k_n},$$

which tells person i that they have k_i.

Thus, we can ensure that any given finite set of us is correct.

Can we do better?

A better solution

Yes!

A better solution

Yes!

We can do much better than finitely correct.

In fact, with the right strategy we can ensure that we ALL guess correctly! Without exception.

Midnight ball solution

We will use the axiom of choice.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

Midnight ball solution

We will use the axiom of choice.

Before the ball, we consider all the conceivable patterns of numbers that we might get.

Two patterns are equivalent, if only finite differences.

Let's agree on a choice of representatives from each class.

At the ball, the queen will choose the total deviation of actual pattern from representative.

We can each see who is correct and who differs from representative (except ourself).

Knowing total difference, we can calculate our number.

We will all be correct!

Generalizations

The solution works even if we have uncountably many guests—any cardinality is fine.

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Solution works with real numbers, instead of natural numbers.

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Solution works with real numbers, instead of natural numbers.

Solutions works with labels from any group, of any size.

Therefore, we can succeed with labels coming from any nonempty set.