A potentialist conception of ultrafinitism

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Ultrafinitism: Physics, mathematics, philosophy
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In some cases, we can seem to say a lot about the properties and features of those numbers.

But in other cases, we are at a loss.

For example, is $e^{e^{e^{79}}}$ an integer? (Skewe's number)

We don't know.

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Ultrafinitism rather should be specifically about the difference in the nature of existence of small and large numbers.

Nevertheless, one has the sense of an affinity between those positions and the ultrafinitist attitude toward very large numbers.

Ultrafinitists on very large numbers

Harvey Friedman raised the “draw the line” objection with ultrafinitist Yessenin-Volpin, concerning existence of

$$2^1, 2^2, 2^3, \ldots, 2^{100}$$

I then proceeded to start with 2^1 and asked him whether this is “real” or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. Sure, he was prepared to always answer yes, but he was going to take 2^{100} times as long to answer yes to 2^{100} then he would to answering 2^1. There is no way that I could get very far with this.

H. Friedman [Fri02, p. 4–5]

Nature of ultrafinitism

The anecdote illustrates the core ultrafinitist idea that number existence becomes increasingly in question as numbers get larger. Numbers become increasingly inaccessible and subject to feasibility issues.

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But the delay-tactic exchange also illustrates what is often seen as a key weakness of ultrafinitism, namely, the lack of a satisfactory formal theory.

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A central difficulty

Problems can arise for an ultrafinitist theories, if the meta-theory is not also ultrafinitist.

Any first-order theory in the language of arithmetic, after all, can form terms $1 + 1 + \cdots + 1$ of unimaginable size.

So we seem to want ultrafinitism already in the metatheory.

The circularity criticism

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My point

The circularity criticism seems to hit both classical and ultrafinitist arithmetic.

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- PA^-, Q, and many others. Rich assortment of theories.
- Finite arithmetic. Arithmetic with a largest number.

A theory of finite arithmetic FA

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- Full induction scheme

$$\left[\varphi(0) \text{ and } \forall n < N \left(\varphi(n) \to \varphi(n+1)\right)\right] \to \forall n\, \varphi(n).$$

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Truncate any model of PA, including nonstandard models, at some number N. (Actually, $I\Delta_0$ suffices.)

Opens up the possibility of ultrafinitist nonstandard analysis.

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A smaller cut with total arithmetic

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Up to b, we can add and multiply freely—it is totally defined.

Base b arithmetic

Work in model M of finite arithmetic FA with largest number N.

Get largest b for which b^2 exists.

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Let's work in base b with, say, five-digit numbers $ABCDE$, with all digits below b.

With grade-school algorithms, we can define how to add and multiply such numbers.

The point: to add and multiply in base b, you only need to know how to add and multiply the digits individually.

So inside M we can add and multiply numbers ABCDE.

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The numbers ABCDE go up to b^5, which exceeds N^2.

We can show M^+ is also a model of FA. Induction still works.

Conclusion

Every model of finite arithmetic, with largest number N, interprets a taller model of finite arithmetic in which N^2 exists.

The two models M and M^+ are bi-interpretable.

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Begin with any model $M_0 \models \mathrm{FA}$.

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Induction in FA becomes bounded induction in M^*.

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So the two versions of ultrafinitism are closely related.

Conclusion (Jeff Paris)

Every model of finite arithmetic arises by truncating a model of $I\Delta_0$ at a number N and conversely.

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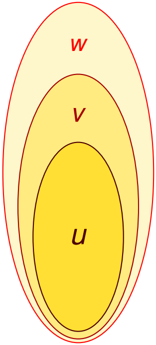
The ultrafinitist can generally agree, however, with the move from M to M^+.

Thus is revealed a potentialist understanding of ultrafinitism.

Potentialism via realms of feasibility

"You can have more and more…"



Consider the realms that are possible: a Kripke model of possible worlds.

A modal perspective on potentialism

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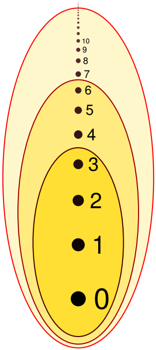
The various universe fragments can be seen as possible worlds in a potentialist system, giving rise to the modal vocabulary:

- $\Diamond\varphi$, if φ holds in some larger world
- $\Box\varphi$, if φ holds in all larger worlds

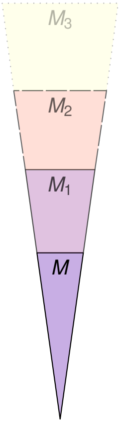
Aristotelian potentialism



Possible worlds consist of all numbers up to some N.

$$u = \{ 0, 1, 2, \ldots, N \}$$

Linear inevitabilism



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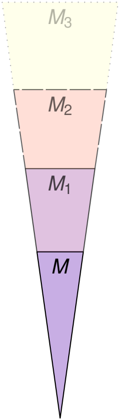

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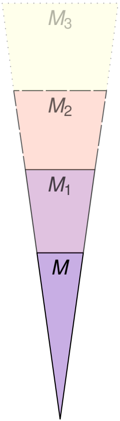

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$$\Diamond\Box\varphi \to \Box\Diamond\varphi$$

But also:

$$\Diamond\varphi \wedge \Diamond\psi \to [\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\varphi)]$$

S4.3 is valid.

Nonlinear potentialism

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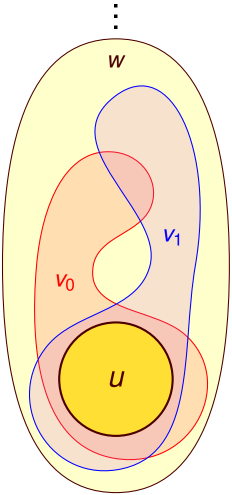
A potentialist might want to say that this number comes into existence before some of the smaller numbers.

Arbitrary set potentialism



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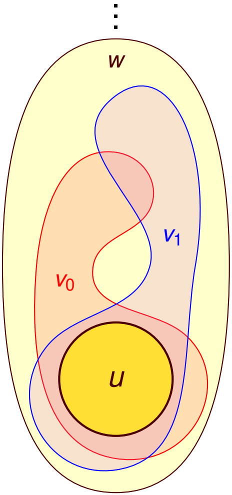

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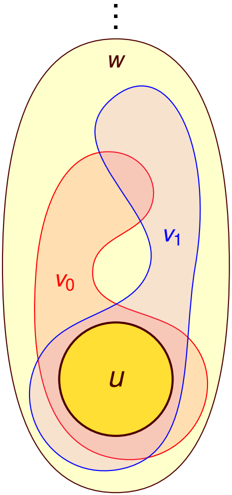

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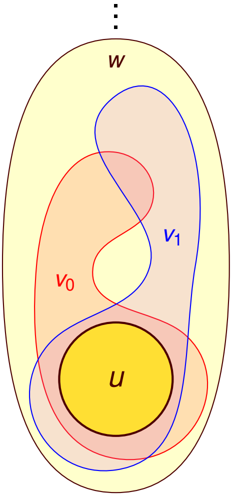

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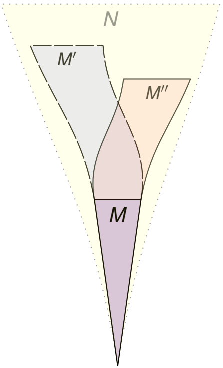
This variety of potentialism exhibits different modal validities.

Convergent potentialism



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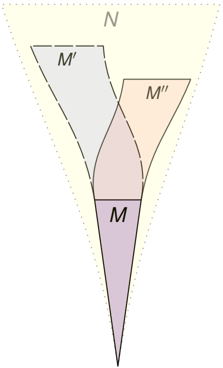

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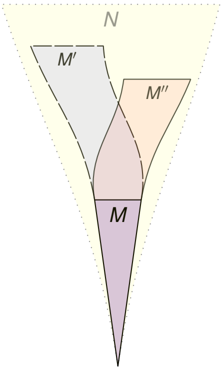

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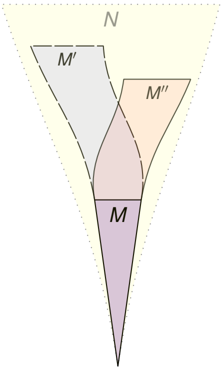

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S4.2 is valid, but perhaps not S4.3.

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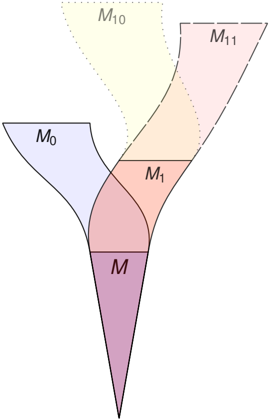
They commit to a determinate underlying nature for the limit of the convergent system.

Truths of the limit model are expressible in the potentialist language and ontology via the potentialist translation:

- Replace $\exists x$ with $\Diamond \exists x$
- Replace $\forall x$ with $\Box \forall x$.

Truth in the limit model thereby becomes expressible in the partial worlds.

Radical branching potentialism



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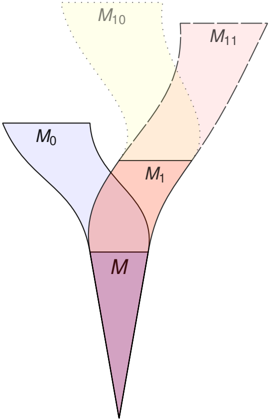

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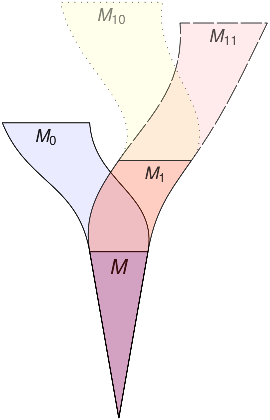

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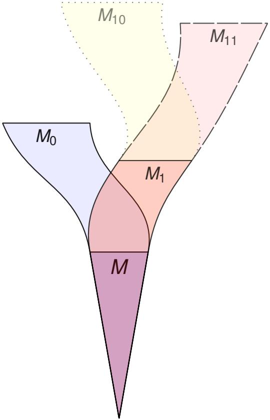

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If a computation is revealed to have output 0, it will never subsequently have output 1, even if that had been possible before.

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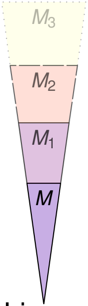
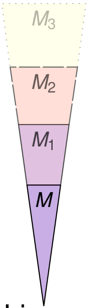
The proof makes use the universal algorithm and control statement method.

Varieties of potentialism

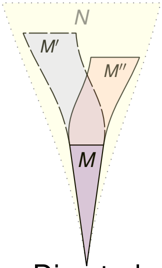


Linear inevitability
S4.3

Varieties of potentialism

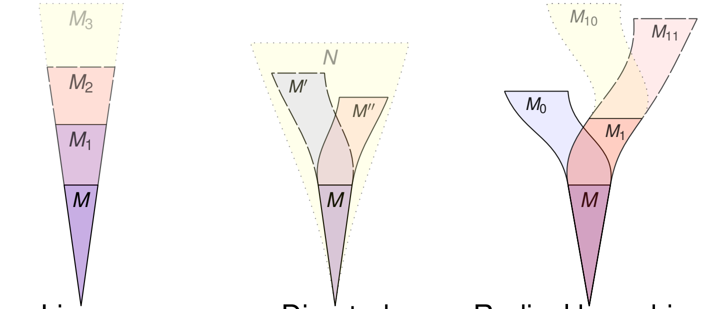


Linear inevitability
S4.3



Directed convergence
S4.2

Varieties of potentialism



Linear inevitability
S4.3

Directed convergence
S4.2

Radical branching possibility
S4

Ultrafinitism as radical-branching potentialism

I find strong affinity ultimately between the ultrafinitist position and the radical branching form of arithmetic pluralism.

Ultrafinitism as radical-branching potentialism

I find strong affinity ultimately between the ultrafinitist position and the radical branching form of arithmetic pluralism.

Friendly suggestion for ultrafinitism

Let's express ultrafinitist ideas in a manner that engages more fully with its affinity with potentialism that I have identified, especially in the radical branching case.

For example, I propose the theory $\Box\mathrm{FA} + \mathrm{S4}$.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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References I

[BE17] Rasmus Blanck and Ali Enayat. "Marginalia on a theorem of Woodin". *J. Symb. Log.* 82.1 (2017), pp. 359–374. ISSN: 0022-4812. DOI: 10.1017/jsl.2016.8.

[Bla17] Rasmus Blanck. "Contributions to the Metamathematics of Arithmetic. Fixed points, Independence, and Flexibility". PhD thesis. University of Gothenburg, 2017. ISBN: 978-91-7346-917-3. http://hdl.handle.net/2077/52271.

[Fri02] Harvey Friedman. "Philosophical problems in logic". seminar notes at the Princeton Philosophy Department, 107 pages. 2002. https://u.osu.edu/friedman.8/files/2014/01/Princeton532-1pa84c4.pdf.

[Ham03] Joel David Hamkins. "A simple maximality principle". *Journal of Symbolic Logic* 68.2 (2003), pp. 527–550. ISSN: 0022-4812. DOI: 10.2178/jsl/1052669062. arXiv:math/0009240[math.LO]. http://wp.me/p5M0LV-2v.

[Ham18] Joel David Hamkins. "The modal logic of arithmetic potentialism and the universal algorithm". *Mathematics arXiv* (2018), pp. 1–35. arXiv:1801.04599[math.LO]. http://wp.me/p5M0LV-1Dh.

References II

[HL08] Joel David Hamkins and Benedikt Löwe. "The modal logic of forcing". *Trans. AMS* 360.4 (2008), pp. 1793–1817. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-07-04297-3. arXiv:math/0509616[math.LO]. http://wp.me/p5M0LV-3h.

[HL13] Joel David Hamkins and Benedikt Löwe. "Moving up and down in the generic multiverse". *Logic and its Applications, ICLA 2013 LNCS* 7750 (2013). Ed. by Kamal Lodaya, pp. 139–147. DOI: 10.1007/978-3-642-36039-8_13. arXiv:1208.5061[math.LO]. http://wp.me/p5M0LV-od.

[HL22] Joel David Hamkins and Øystein Linnebo. "The modal logic of set-theoretic potentialism and the potentialist maximality principles". *Review of Symbolic Logic* 15.1 (2022), pp. 1–35. ISSN: 1755-0203. DOI: 10.1017/S1755020318000242. arXiv:1708.01644[math.LO]. http://wp.me/p5M0LV-1zC.

References III

[HLL15] Joel David Hamkins, George Leibman, and Benedikt Löwe. "Structural connections between a forcing class and its modal logic". *Israel Journal of Mathematics* 207.2 (2015), pp. 617–651. ISSN: 0021-2172. DOI: 10.1007/s11856-015-1185-5. arXiv:1207.5841[math.LO]. http://wp.me/p5M0LV-kf.

[HW17] Joel David Hamkins and W. Hugh Woodin. "The universal finite set". *Mathematics arXiv* (2017), pp. 1–16. arXiv:1711.07952[math.LO]. http://jdh.hamkins.org/the-universal-finite-set.

[Lin13] Øystein Linnebo. "The Potential Hierarchy of Sets". *Review of Symbolic Logic* 6.2 (2013), pp. 205–228.

[LS17] Øystein Linnebo and Stewart Shapiro. "Actual and Potential Infinity". *Noûs* (2017). ISSN: 1468-0068. DOI: 10.1111/nous.12208.

[Woo11] W. Hugh Woodin. "A potential subtlety concerning the distinction between determinism and nondeterminism". In: *Infinity*. Cambridge University Press, 2011, pp. 119–129.