A Potentialist Perspective on Ultrafinitism

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This talk is based in part on my article:

Joel David Hamkins. "A Potentialist Conception of Ultrafinitism". *Philosophia Mathematica* (2026). To appear. Adapted from my talk at the Ultrafinitism Conference at Columbia University, April 2025. arXiv:2512.06564[math.LO]

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See also my essay series on Infinitely More:

https://infinitelymore.xyz/t/ultrafinitism

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In some cases, we can seem to say a lot about the properties and features of those numbers.

But in other cases, we are at a loss.

For example, is $e^{e^{e^{79}}}$ an integer? (Skewe's number)

We don't know.

Ultrafinitist ontology

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Ultrafinitism rather should be specifically about the difference in the nature of existence of small and large numbers.

Nevertheless, one has the sense of an affinity between those positions and the ultrafinitist attitude toward very large numbers.

Ultrafinitists on very large numbers

Harvey Friedman raised the “draw the line” objection with ultrafinitist Yessenin-Volpin, concerning existence of 2^{100}.

$$2^1, 2^2, 2^3, \ldots, 2^{100}$$

> *I then proceeded to start with 2^1 and asked him whether this is "real" or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. Sure, he was prepared to always answer yes, but he was going to take 2^{100} times as long to answer yes to 2^{100} then he would to answering 2^1. There is no way that I could get very far with this.*
>
> *H. Friedman [Fri02, p. 4–5]*

Nature of ultrafinitism

The anecdote illustrates the core ultrafinitist idea that number existence becomes increasingly in question as numbers get larger. Numbers become increasingly inaccessible and subject to feasibility issues.

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But the delay-tactic exchange also illustrates what is often seen as a key weakness of ultrafinitism, namely, the lack of a satisfactory formal theory.

What is ultrafinitism exactly?

Can we present a formal theory of ultrafinitism, say, as a theory in first-order logic?

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A central difficulty

Problems can arise for an ultrafinitist theories, if the meta-theory is not also ultrafinitist.

Any first-order theory in the language of arithmetic, after all, can form terms $1 + 1 + \cdots + 1$ of unimaginable size.

Some formalizations of ultrafinitism adopt measures in effect to block this kind of move.

Ultrafinitists thus often seek to have ultrafinitism in the metatheory.

The circularity criticism

If metatheoretic commitments are taken as logically prior to the object theory, then by working in an ultrafinitist metatheory we seem to need already to know what ultrafinitism is (in the metatheory) before saying what it is (in the object theory).

To my way of thinking, this is a kind of circularity problem.

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My point

The circularity criticism seems to hit both classical and ultrafinitist arithmetic. Judgement: standstill.

A hierarchy of extremely weak theories

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- Finite arithmetic. Arithmetic with a largest number.

Two views of ultrafinitism

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- Some forms of ultrafinitism postulate a realm of feasible numbers, closed under successor, but not closed under exponentiation. Strong forms explicitly deny 2^{100}.
- Another totally different form of ultrafinitism postulates the existence of a largest number.

Realm of feasible numbers

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And yet, the totality of exponentiation is denied.

At bottom, addition and multiplication are taken as innocent, while exponentiation is not.

Bounded induction $I\Delta_0$ as a form of ultrafinitism

The theory of bounded induction $I\Delta_0$ implements this perspective quite well, and in this sense, we may take it as a form of ultrafinitism.

The theory $I\Delta_0$ proves that addition and multiplication are total functions, but it does not prove that exponentiation is total.

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Meanwhile, $\mathrm{I}\Delta_0$ does prove that 2^{100} exists.

Arithmetic with a largest numbers

There is another totally different approach to ultrafinitism.

The theory of Finite Arithmetic FA asserts that there is a largest number.

Let us discuss the details. This theory is also known as PA^{top}.

A theory of finite arithmetic FA

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- Full induction scheme

$$\left[\varphi(0) \text{ and } \forall n < N\,(\varphi(n) \to \varphi(n+1))\right] \to \forall n\, \varphi(n).$$

Models of finite arithmetic

Standard finite models $\mathbb{N} \upharpoonright n = \{0, 1, \ldots, n\}$ for any standard finite n.

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We shall have nonstandard models as well. e.g. Ultraproducts of these models. Or get them by compactness, etc.

Truncate any model of PA, including nonstandard models, at some number N. (Actually, $I\Delta_0$ suffices.)

Opens up the possibility of ultrafinitist nonstandard analysis.

A criticism

Something unsettling about the theory of finite arithmetic.

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The ad hoc criticism

Why stop at N? What could be the justification for stopping at any particular point?

If you made it that far, couldn't you go a little further?

I should like to explore this criticism. I want to give legs to this criticism.

A smaller cut with total arithmetic

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Basically, b is $\sqrt{N}$, rounded up.

Below b, we can add and multiply freely—it is totally defined.

Base b arithmetic

Work in model M of finite arithmetic FA with largest number N.

Get smallest b for which b^2 does not exist.

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Get smallest b for which b^2 does not exist.

Let's work in base b with, say, five-digit numbers $ABCDE$, with all digits below b.

We consider these 5-tuples of digits as a kind of name for an imaginary number, which might not itself exist in M.

Grade-school algorithms

With grade-school algorithms, we can define how to add and multiply such numbers.

```
    ABCDE
+   FGHIJ
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    PQRST
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Key point: to add and multiply in base *b*, you only need to know how to add and multiply the digits individually.

So inside *M* we can add and multiply numbers ABCDE. Imaginary numbers in *M*, without meaning.

Interpreting a taller model

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The numbers ABCDE go up to $b^5 - 1$, which exceeds N^2.

We can show M^+ is also a model of FA. Induction still works.

Conclusion

Every model of finite arithmetic, with largest number N, interprets a taller model of finite arithmetic in which N^2 exists.

The two models M and M^+ are bi-interpretable.

Supporting the ad hoc criticism

I take the interpretation to give teeth to the ad hoc criticism.

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Finite arithmetic FA purports to give an account of arithmetic by providing a realm of numbers from 0 up to N.

But using only those ontological resources, allowing pairs or 5-tuples, we can interpret a much taller system.

So there was no need to stop at N.

We can iterate the construction

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The limit model

$$M^* = \bigcup_n M^{+n}$$

has $+$ and $\cdot$ totally defined. Arithmetic becomes total.

Induction in FA becomes bounded induction in M^*.

Connection between Finite arithmetic and $I\Delta_0$

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So the two versions of ultrafinitism are closely related.

Conclusion (Jeff Paris)

Every model of finite arithmetic arises by truncating a model of $I\Delta_0$ at a number N and conversely.

Potentialist nature of ultrafinitism

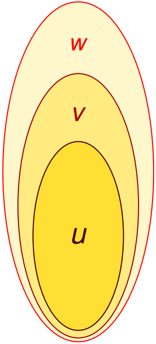
The ultrafinitist, of course, does not agree with the M^* construction. That's fine.

The ultrafinitist can generally agree, however, with the move from M to M^+.

Thus is revealed a potentialist understanding of ultrafinitism.

Potentialism via realms of feasibility

"You can have more and more. . . "



Consider the realms that are possible: a Kripke model of possible worlds.

A modal perspective on potentialism

Current philosophical work emphasizes the *modal* nature of potentialism.

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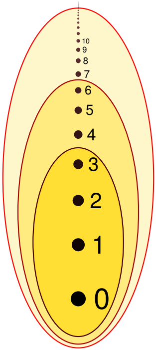
The various universe fragments can be seen as possible worlds in a potentialist system, giving rise to the modal vocabulary:

- $\Diamond\varphi$, if φ holds in some larger world
- $\Box\varphi$, if φ holds in all larger worlds

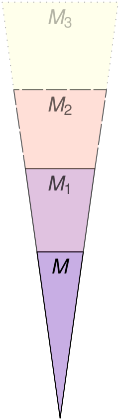
Aristotelian potentialism



Possible worlds consist of all numbers up to some N.

$$u = \{\, 0, 1, 2, \ldots, N \,\}$$

Linear inevitabilism



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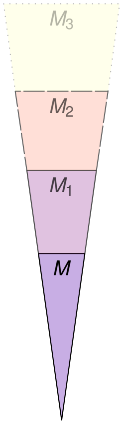

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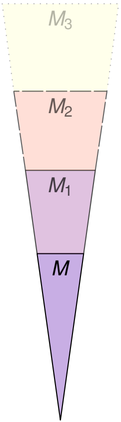

The possible worlds are building up to a limit world in a linear coherent manner.

Every possibly necessary assertion is also necessarily possible.

$$\Diamond\Box\varphi \to \Box\Diamond\varphi$$

But also:

$$\Diamond\varphi \wedge \Diamond\psi \to [\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\varphi)]$$

S4.3 is valid.

Nonlinear potentialism

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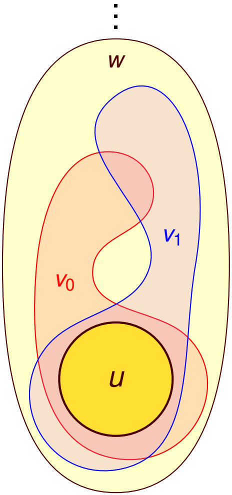
A potentialist might want to say that this number comes into existence before some of the smaller numbers.

Arbitrary set potentialism



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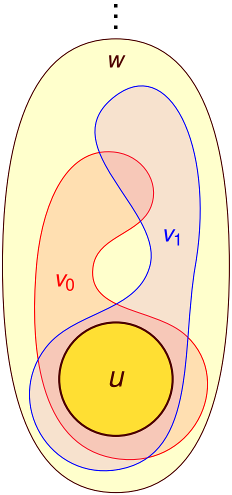

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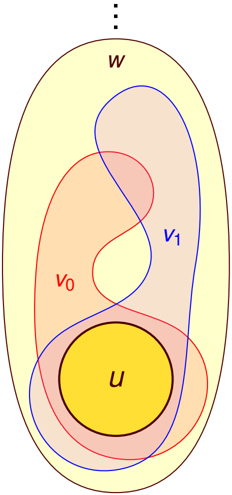

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A reasonable, but fundamentally different perspective on potentialism.

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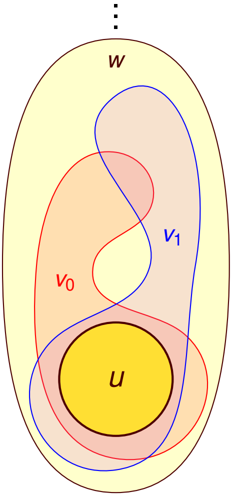

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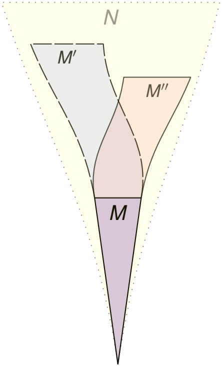
This variety of potentialism exhibits different modal validities.

Convergent potentialism



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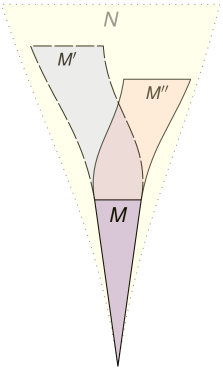

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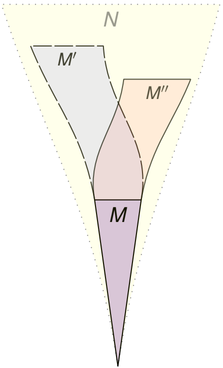

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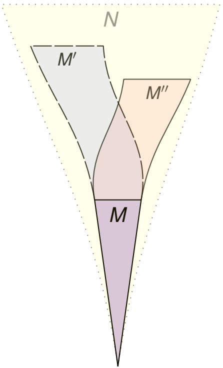

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S4.2 is valid, but perhaps not S4.3.

Implicit actualism

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Implicit actualism

In my view, the linear and convergent forms of potentialism are not genuinely potentialist, but implicitly actualist.

They commit to a determinate underlying nature for the limit of the convergent system.

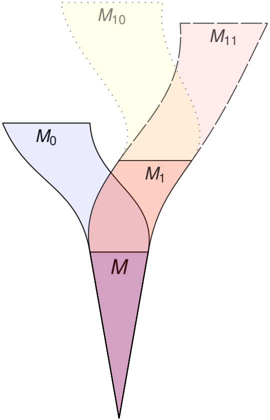
Truths of the limit model are expressible in the potentialist language and ontology via the potentialist translation:

$$\psi \quad \mapsto \quad \psi^{\Diamond}$$

- Replace $\exists x$ with $\Diamond\exists x$
- Replace $\forall x$ with $\Box\forall x$.

Truth in the limit model thereby becomes expressible in the partial worlds.

Radical branching potentialism



A more radical form of potentialism.

What is possible/necessary may depend on what's already happened.

As objects become actual, they may close off some alternative possibilities.

If a computation is revealed to have output 0, it will never subsequently have output 1, even if that had been possible before.

Models of PA under end-extension

Exactly this situation is realized in the models of PA under end-extension. We view each (nonstandard) model of PA as a realm of feasibility.

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Moving to a larger realm is to invoke the possibility operator.

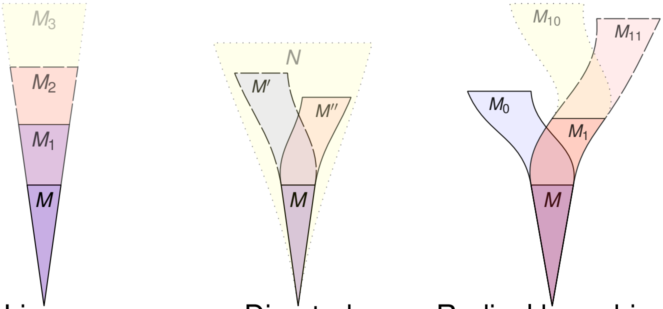
Theorem (Hamkins [Ham26b])

Arithmetic end-extensional potentialism exhibits radical branching, and the modal validities are exactly only S4.

Thus, taking nonstandard models of PA as realms of feasibility, we achieve the radical-branching form of potentialism.

The proof makes use the universal algorithm and control statement method.

Varieties of potentialism



Linear inevitability	Directed convergence	Radical branching possibility
S4.3	S4.2	S4

Ultrafinitism as radical-branching potentialism

I find strong affinity ultimately between the ultrafinitist position and the radical branching form of arithmetic pluralism.

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Friendly suggestion for ultrafinitism

Let's express ultrafinitist ideas in a manner that engages more fully with its affinity with potentialism that I have identified, especially in the radical branching case.

For example, I propose the theory $\Box\mathrm{FA} + \mathrm{S4}$.

Inevitability

New work in progress, joint with Chris Peng (Notre Dame).

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We consider the collection of models of finite arithmetic FA as a potentialist system.

We seek to find the assertions that are *inevitable* in this system.

Inevitability

An assertion ψ is *inevitable* with respect to a potentialist system, if every model satisfies the potentialist translation $\psi^{\Diamond}$.

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The *potentialist translation* $\psi^{\Diamond}$ of a formula ψ replaces all instances of $\exists x$ with $\Diamond\exists x$ and all instances of $\forall x$ with $\Box\forall x$.

The potentialist translation is important in convergent potentialist systems. When system $\mathcal{M}$ converges to limit model M, then

$$M \models \psi \quad \text{if and only if} \quad W \models_{\mathcal{M}} \psi^{\Diamond}.$$

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The potentialist translation is important in convergent potentialist systems. When system $\mathcal{M}$ converges to limit model M, then

$$M \models \psi \quad \text{if and only if} \quad W \models_{\mathcal{M}} \psi^\Diamond.$$

Actualist truth reduces to the potentialist ontology.

This was part of my charge that convergent potentialism is "implicitly actualist."

Inevitability in FA

But FA is not convergent. There is no limit model. It has radical branching.

Inevitability in FA

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Nevertheless, we can seek to understand which assertions are inevitable in this system.

Question

Which assertions are inevitable with respect to finite arithmetic FA?

Inevitable over FA

We saw that for every model M of FA, there is a taller model M^+ in which $a+b$ and ab exist for all $a, b \in M$.

So the assertion that addition and multiplication are total is inevitable with respect to FA.

$$\Box \forall a, b \Diamond \exists c, d \quad c = a + b \quad \wedge \quad d = ab.$$

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$$\Box \forall a, b \Diamond \exists c, d \quad c = a + b \quad \wedge \quad d = ab.$$

What else do we get?

Inevitable over FA

Peng and I have proved that $I\Delta_0$ is inevitable in the FA potentialist system.

Theorem

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Inevitable over FA

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Main ideas:

- The partiality of the function terms causes subtle and confusing problems.
- Every FA model extends to higher FA models in which the arithmetic is total on the prior individuals.
- Every FA model can validate bounded induction in the taller FA models that it interprets.

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Main ideas:

- The partiality of the function terms causes subtle and confusing problems.
- Every FA model extends to higher FA models in which the arithmetic is total on the prior individuals.
- Every FA model can validate bounded induction in the taller FA models that it interprets.
- Every instance of Δ_0 induction will continue further to all FA extensions.

Conjecture

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The theory $I\Delta_0$ is exactly what is inevitable with respect to FA.

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The theory $I\Delta_0$ is exactly what is inevitable with respect to FA.

This is philosophically interesting, because the FA potentialist system is not convergent, and so not subject to prior mirroring results.

To my knowledge, this would be the first instance of a summative inevitability result in the absence of S4.2, and without the existence of a limit model toward which the potentialist system is converging.

Coming full circle

We began with two extremely different approaches to ultrafinitism:

- Bounded arithmetic $I\Delta_0$ limits induction, keeps addition and multiplication, but blocks exponentiation.
- Finite arithmetic FA asserts there is a largest number.

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- Bounded arithmetic $I\Delta_0$ limits induction, keeps addition and multiplication, but blocks exponentiation.
- Finite arithmetic FA asserts there is a largest number.

But ultimately, we revealed deep model-theoretic connections between these ideas.

- Every model of FA interprets taller models that converge to a unique minimal extension model of $I\Delta_0$
- The theory $I\Delta_0$ is inevitable over FA.
- Conjecture: exactly this theory is inevitable over FA.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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